THE

MATHEMATICAL GAZETTE

EDITED BY

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THE SUM AND PRODUCT FUNCTIONS.

By V. Naylor.

If we keep an eye on the many *interpretations* of the operations of multiplication and addition rather than on the actual *operations* themselves, we shall need to make several changes of focus.

Let us see what these *operations* could mean without being concerned with what their *interpretations* could be.

The expressions "*sum of*", "*product of*" are expressions of functionality.

Accordingly, the "sum of" two symbols a and b is a function of a and b which we may denote by $S(a, b)$; $S(a, b)$ is a function of two independent variables a and b.

We shall not introduce a new symbol for this functionality.

Likewise, the "product of" a and b is a function of a and b which we shall denote by $P(a, b)$.

The rules for using $S(a, b)$ and $P(a, b)$ are:

The Associative Rule.

(1) $S\{S(a, b), c\}=S\{a, S(b, c)\}$. (1′) $P\{P(a, b), c\}=P\{a, P(b, c)\}$.

The Commutative Rule.

(2) $S(a, b)=S(b, a)$. (2′) $P(a, b)=P(b, a)$.

The Distributive Rule.

(3) $P\{a, S(b, c)\} =S\{P(a, b), P(a, c)\}$. (3′) $P\{S(b, c), a\} =S\{P(b, a), P(c, a)\}$.

The Identity Operator.

(4) The identity symbol for S is 0 and is defined by $S(a, 0)=a=S(0, a)$.*

(4′) The identity symbol for P is 1 and is defined by $P(a, 1)=a=P(1, a)$.

* i.e. that symbol b which makes $S(a, b)=a$.

The result of using 0 *in connection with* P.

By virtue of (2′), (3) and (3′) are the same; they form the connecting link between S and P.

In (3), put $c=0$,

$$P\{a, S(b, 0)\}=S\{P(a, b), P(a, 0)\},$$

$$P\{a, b\}=S\{P(a, b), P(a, 0)\} \quad \text{(by 4)},$$

$$P(a, 0)=0 \quad \text{(again by 4)},$$

and $$P(0, a)=0 \quad \text{(by 2′)}.$$

This means that the zero of addition (*i.e.* as defined at 4) has the usual property when used in connection with multiplication, viz.:

$$a\times 0=0=0\times a.$$

The result of using 1 *in connection with* S.

Putting $a=1$ in (4), we get

$$S(1, 0)=1=S(0, 1) \quad \ldots\ldots\ldots\ldots (\beta)$$

We use this to form the natural numbers. Thus we write

$$2, 3, 4, \ldots$$

as new symbols for

$$S(1, 1),\ S(1, 2),\ S(1, 3), \ldots .$$

So that $S(1, 1)=2$ may be written as

$$S\{1, S(1, 0)\}=2, \quad \ldots\ldots\ldots\ldots \text{(by } \beta)$$

or $$S1\,S1, 0=2$$

when brackets are omitted.

Likewise $S(1, 2)=3$ may be written as

$$S[1, S\{1, 1\}]=3,$$

or $$S[1, S\{1, S(1, 0)\}]=3,$$

or $$S1\,S1\,S1, 0=3.$$

Likewise $S(1, 3)=4$ gives $S1\,S1\,S1\,S1, 0=4$, etc.

Those who need a sign may, if they like, replace S by $+$.

But, since $S(a, b)$ may equally well be written as $(a, b)\,S$, it is immaterial where this sign is placed. We have assumed that the brackets play no essential part in the reasoning; this is obvious from the fact that $S(a, b)$ may be written as Sab.

In fact all functions which behave like P and S, I should call products and sums.

For example, the couple $(xx'-yy', x'y+xy')$ is called the *product* of the two couples (x, y), (x', y') because, and only because, the first is the same function of the second and third that $P(a, b)$ is of a and b; (x, y) is not an ordinary number.

For similar reasons, 3 feet is called a product.

It now only remains to show how these "quantitative" integers

2, 3, 4, ... behave when used "operationally". This is done by a simple example.

To show that $2\times a=a+a$.

In (3) put $b=1$, $c=1$; we get

$$P\{a, S(1, 1)\}=S\{P(a, 1), P(a, 1)\},$$
$$P\{a, S(1, 1)\}=S\{a, a\} \quad \text{(by 4')},$$
$$P\{a, 2\}=S\{a, a\} \quad \text{for } S(1, 1)=2.$$

This proves the result usually written as $2\times a=a+a$. It identifies the "quantitative 2" with the "operational 2".

The results of the foregoing work are, as stated at the beginning, capable of many interpretations, each interpretation seeming to suggest a natural place for the symbols of functionality P and S(or $\times$ and $+$). But as we have seen above, precedence is given to neither a nor b in the definitions of P and S.

Similar remarks can be made about the concept of division. Such word expressions as "the ratio of a to b", "the division of a by b", "the measure of a, b being the unit", "the quotient of a and b", etc., are synonymous. They all stand for the one idea $Q(a, b)$ where Q is a function which obeys well-known and clearly defined laws. They owe their origin to the various practical needs which have been met by the quotient function at the various stages in its development.

V. NAYLOR.

EDINBURGH MATHEMATICAL SOCIETY.

UNDER the auspices of the Edinburgh Mathematical Society, a Mathematical Colloquium will be held in St. Andrews, Scotland, from July 18th to 28th, 1934. The following lecture courses have been arranged: (*a*) "World-Structure by the Kinematic Methods of the Special Theory of Relativity" by Prof. E. A. Milne (Oxford); (*b*) "Ramanujan's Note-Books and their place in Modern Mathematics" by Prof. B. M. Wilson (Dundee); (*c*) "Pictorial Geometry" by Prof. H. W. Turnbull (St. Andrews), and (*d*) "Expansions relating to the Problem of Lattice Points" by Mr. W. L. Ferrar (Oxford). In addition there will be single lectures and informal discussions.

By the courtesy of the St. Andrews University Court, the Colloquium will be held in University Hall. The cost of board and lodging in the Hall will be £5 5s. 0d. per head, and the combined fee for lectures will be £1 5s. 0d. Early application should be made by those wishing to stay in the Hall as the accommodation is limited.

A number of well-known mathematicians have announced their intention of being present, and the chief foreign guest of the Society will be the eminent Dutch astronomer, Prof. W. de Sitter.

Membership forms can be obtained from Dr. G. C. McVittie, Colloquium Secretary, 16 Chambers Street, Edinburgh, Scotland.

DIFFERENTIALS.*

Prof. G. Temple (King's College, London) :

THE THEORY OF DIFFERENTIALS.

The Problem.

In the teaching of the Theory of Differentials the really difficult and important problem is *not* " how shall we teach a given theory ? " but " which theory shall we teach ? " There are, in fact, a number of rival theories, none of which has met with any general measure of acceptance. In these circumstances, the critical problem of considering the rival merits of these theories is much more urgent than the pedagogical problem of presenting them to students.

I propose, therefore, to give a brief survey of three rival theories of differentials, and in doing so I shall limit myself to those theories of differentials which are logically prior to and independent of the theory of the functional derivative. I exclude from consideration those theories of differentials which depend upon the theory of the functional derivative, first, because I have nothing to add to the account of this subject which is to be found in any *Traité d'Analyse*, and, secondly, because the only value of such a theory lies in the justification of the notation which separates the differentials in a derivative. Whereas such theories as these require a preliminary discussion of the functional derivative, the theories which I review below allow the functional derivative to be defined as a ratio of two differentials.

The Three Possible Forms of a Theory of Differentials.

The three possible forms which can be taken by a theory of differentials can easily be discovered by the following elementary considerations.

The problem before us is to frame a definition of the infinitesimal number η in such a way that either

(*a*) the functional derivative $f'(x)$ is equal to the fraction

$$[f(x+\eta)-f(x)]/\eta,$$

or (*b*) $$\eta f'(x)=[f(x+\eta)-f(x)].\dagger$$

It is sufficient to consider the particular case $f(x)=x^2$, when the preceding relations become

(*a*) $$2x=[2x\eta+\eta^2]/\eta$$

or (*b*) $$2x\eta=2x\eta+\eta^2.$$

These relations may be satisfied in three ways. Either

(1) $$\eta^2=0, \quad \text{but} \quad \eta\neq 0\,;$$

or (2) $$\eta=0\,;\ddagger$$

* A discussion at the Annual Meeting, 5th January, 1934.

† The second condition is more general than the first, for it applies even when η has no reciprocal, as in Case I below.

‡ " There are zeros and zeros ! " (O. Heaviside).

or (3) some special meaning is given to the sign of equality.

The first possibility leads to Kramer's theory of the Dual Variable; the third possibility is realised in Veronese's Theory of Non-Archimedean Numbers, and the second possibility is realised in the theory of nul sequences. The three possibilities will be discussed in this order.

In each case the problem before us is to establish the existence of a "number" η having certain special properties, which distinguish it from numbers belonging to the class of ordinary real numbers. This "existence" theorem is proved by constructing a "representation" of η in terms of ordinary real numbers.

(1) *The Theory of the Dual Variable.*

The theory of the dual variable may be briefly dismissed by noting that a representation of the number satisfying the equations $\eta^2=0$, but $\eta \neq 0$, is provided by the matrix $\eta = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$.

The use of this "dual" number η enables us to define the functional derivative by the equation

$$f(x+h\eta)-f(x)=h\eta f'(x)$$

but the theory is obviously far too artificial to play any effective part in analysis.

(2) *The Theory of Non-Archimedean Numbers.*

According to Veronese the non-Archimedean number η is defined by the condition that $N\eta < 1$ for all positive real numbers N. To obtain a representative of such numbers, consider the algebra of ordered number pairs (x, y) with the following definition of addition and inequality:

(1) $(a, b)+(c, d)=(a+c, b+d)$;

(2) $(a, b) < (c, d)$ if $a < c$, for all b and d,
$(a, b) < (a, d)$ if $b < d$.

Then if we denote $(0, 1)$ by η and $(1, 0)$ by ϕ, it is clear that $N\eta < \phi$ for all positive real numbers N. Here ϕ is the unit of the finite number system and η is the unit of the infinitesimal number system.

If we employ the infinitesimal unit η, we can define the functional derivative $f'(x)$ by the condition that

$$N\left\{\frac{f(x+h\eta)-f(x)}{h\eta}-f'(x)\right\}<1$$

for all N.

Here again we shall not linger over this theory, as it will probably seem repellent and artificial to most students and teachers.

(3) *The Theory of Nul Sequences.*

The following theory of differentials is a modification and extension of the Weierstrass-Cantor theory of irrational numbers. The

essence of this theory is the definition of a real number as a convergent sequence of rational numbers. It is natural to define an infinitesimal to be a sequence with the unique limit zero. Such a sequence is called a "nul" sequence.

In this theory two real numbers a and b, represented by the sequences $\{a_n\}$ and $\{b_n\}$ are said to be equal if the sequence $\{a_n - b_n\}$ tends to zero. It is convenient to modify this definition of equality in the case of nul sequences. Two nul sequences $\{a_n\}$ and $\{b_n\}$ will be said to be "equal" if the sequence $\{a_n/b_n\}$ tends to unity.

If we use the symbol η to denote an infinitesimal, *i.e.* a nul sequence, then the value of the functional derivative $f'(x)$ at the point $x=a$ can be defined to be the number $f'(a)$ (if it exists), such that the infinitesimal $f(a+\eta)-f(a)$ "equals" the infinitesimal $\eta f'(a)$.

The differential of $f(x)$ at the point $x=a$ can be defined to be $f(a+\eta)-f(a)$, *i.e.* it is represented by a certain nul sequence.

Criticism.

There is of course no question of discussing which of these theories is "correct"; each theory is self-consistent and leads to a possible representation of derivatives as ratios of differentials.

The only pertinent question is as to which theory is most in accordance with the general spirit of modern analysis. The third theory is clearly a natural extension of the simplest theory of irrational numbers, and there can be little doubt that this theory is in closest relation with the modern methods. The pedagogical question as to whether it is advisable to teach this theory to elementary students is one which I am happy to leave for discussion by subsequent speakers.

References.

I. THE THEORY OF THE DUAL VARIABLE.

E. E. Kramer, *Amer. Journ. Maths.*, 52 (1930), p. 370.

II. THE THEORY OF NON-ARCHIMEDEAN NUMBERS.

G. Veronese, *Atti Accad. Lincei* (*Memorie Mat.*), (4) 6 (1890), p. 603.

Fondamenti di Geometria, Padua (1891), p. 105.

A. Schoenflies, *Jahresb. deutsch. Math.-Ver.*, 15 (1906), p. 26.

Encyc. Sci. Math., III, 1, Fasc. 1 (1911), p. 148.

III. THE THEORY OF NUL SEQUENCES.

E. W. Hobson, *The Theory of Functions of a Real Variable*, 2nd edit., 1 (1926), Chap. 1.

Mr. J. T. Combridge (King's College, London): It seems to me that the best thing I can do this afternoon, in relation to the utility of this discussion as a whole, is to review the present position of the teaching of differentials and to indicate what seem to me to be good or bad lines for the subsequent discussion to follow.

In 1931, in the July number of the *Mathematical Gazette*,* there appeared an article by Mr. E. G. Phillips—there are two mathematicians with that name and those initials ; I leave you to discover which was the author—deploring the fact that "frequently it happens that students . . . come up to the university never even having heard of a differential ! " There followed immediately the statement that "it would be of the greatest possible assistance to those who are responsible for the later teaching of the Calculus if the schoolmasters taught the subject from the differential standpoint right from the start ".

That this statement was merely an expression of his own opinion was at once acknowledged by Mr. Phillips in reply to a protest which I entered against it. That protest, together with a note by Professor J. R. Wilton of Adelaide, a reply by Mr. Phillips and a historical note by the Editor, appeared in 1932 in the February number of the *Gazette*.† In it I expressed in italics the view that, however good the teacher, it is not safe to give students any inkling of differentials until they are thoroughly accustomed to dealing with derivatives as exact limits. (I must apologise for the personal note, but it seems to me that if a discussion is not personal it is not worth having.)

In the summer of 1933 there was issued an advance notice of the forthcoming publication of Volume I of an *Elementary Calculus* by two men whose names were a guarantee at once of scholarship and sales. Judge of my horror when I found that differentials were introduced on p. 25 at the beginning of Chap. III. I have since discovered that another new *Elementary Calculus* introduces them on p. 17.

It appeared desirable then that the fullest possible ventilation should be given at the earliest possible moment to this question of the teaching of differentials in schools, in order that both sides should be thoroughly discussed. Professor Temple having once incautiously expressed in my hearing, in one of his more paradoxical moments, the opinion that there were no such things as differentials, it seemed a good idea that we should try to come to some conclusion on the best way and time for introducing them, because their utility in advanced mathematics is beyond dispute. The Programme Committee, with their usual adroitness, called upon Professor Temple for a paper on the subject, and subpoenaed me, as chief complainant, to initiate the subsequent discussion.

Well, you have heard Professor Temple. Let me say at once that my opinion as to the inadvisability of the early teaching of differentials has not been altered by anything that he has said this afternoon. Probably if what he said was the last word on the subject most of you would agree with me. But it is not the last.

It used to be said of Dr. John Henry Newman by his opponents (with what degree of truth this is not the place to discuss) that he

* Vol. XV., p. 401.

† Vol. XVI., p. 7.

would write " a whole sermon, not for the sake of the text or of the matter, but for the sake of one single passing hint—one phrase, one epithet, one little barbed arrow, which as he swept magnificently past on the stream of his calm eloquence, seemingly unconscious of all presences, save those unseen, he delivered unheeded, as with his finger-tip, to the very heart of an initiated hearer, never to be withdrawn again ".*

Now if you followed closely Professor Temple's introduction, you noticed that he mentioned very casually that there is, of course, the treatment of differentials given in such well-known works as de la Vallée Poussin's *Cours d'Analyse*, in which the differentials dy and dx are, in effect, two finite non-zero quantities whose ratio is equal to the derivative of y with respect to x, but he did not purpose dealing with differentials as defined in this method of treatment.

But this is precisely the treatment advocated by Mr. Phillips and inculcated by the authors of the *Elementary Calculus* to which I have referred. Professor Temple's paper has served the purpose it was intended to serve if it confines the discussion to this particular method of treatment, but I wish now to limit the field still further. It is unnecessary to go over again all that has appeared in the *Gazette*; it will be enough for clearness if I recall a couple of definitions:

(1) " A function $f(x)$ is said to be *differentiable* at the point x if it is finite and determinate in the neighbourhood of the point, and if, when x is given an arbitrary increment Δx, the corresponding increment Δy can be expressed as the sum of two terms:

$$\Delta y = A \,.\, \Delta x + \epsilon \,.\, \Delta x,$$

A being independent of Δx and ϵ tending to zero with Δx."

(2) " Then the first term, which is simply proportional to Δx, is called the *differential* of y and is denoted by dy." †

It is easily seen, by putting $y = x$, that $dx = \Delta x$.

Now the kind of controversial topic which I should like to see excluded this afternoon is that on which Professor Wilton and Mr. Phillips engaged when they began to discuss whether or not the statement " ϵ tends to zero as Δx tends to zero in the above equation *as it stands* " is meaningless. That kind of controversy is best conducted on paper; certainly not by people limited in time as we are. I think we should for the moment take the question before us to be this:

Assuming these definitions and the associated method of treatment to be accepted by quite a number of reputable mathematicians, is it desirable that they should be a recognised part of an elementary school course in calculus?

The answer which I personally would give to this question is still " No ", and for these reasons:

* J. H. Newman, *Apologia pro vita sua* (1908), p. 311.

† de la Vallée Poussin, *Cours d'Analyse Infinitésimal*, I, (1923), p. 51, or E. G. Phillips, *Math. Gazette*, XV, (1931), p. 402.

(1) The importance of distinguishing between derivatives and differential coefficients arises, and can be best understood, in dealing with functions of two or more variables.

(2) Uses of so-called differentials when only one independent and one dependent variable are involved are matters of convenience (or often laziness) in notation, and can generally be avoided. The chief difficulty arises from our not having in common use a sign such as $f^{-1}(x)$ to express $\int(x)\,dx$ in the same way as $f'(x)$ expresses $df(x)/dx$.

(3) The pupil who is ultimately to reach a stage in mathematics at which he will meet with functions of more than one variable may, in common with his less mathematical fellows, be given nevertheless a perfectly sound introduction to the calculus by suppressing altogether the term "differential coefficient", and referring only to derivatives.

Perhaps in the course of this discussion someone will be able to give us a reason for introducing differentials at an early stage, other than the desire to give an answer to the question, "Why is this thing called a differential coefficient ?"

It may be objected that it is not practicable to abolish altogether the name "differential coefficient" and that in any case it is not easy to find a verb to replace "differentiate". I agree. My own procedure would be to begin with the differential coefficient as

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x},$$

and to introduce afterwards the derivative as

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

The former is approached most easily by graphical methods, and then subsequently one can point out that the result depends only on the original form of the function of x; the latter, I find from my own teaching experience, is a more difficult concept to begin with than the former. Then one can explain that the name "differential coefficient" is a piece of intelligent anticipation.

Two particular remarks before I finish :

(i) Let no one think that the early introduction of differentials removes the need for introducing the concept of a limit. The derivative is still a limit, and to my own mind the first essential is to get it absolutely clear that when the derivative exists it gives accurately the gradient of the tangent to the curve representing the original function.

(ii) Differentials are not needed in the teaching of errors. If S is the area of a square, the length of whose side is x, the error in the area due to taking $x+\Delta x$ as the length of the side is exactly

$$2x(\Delta x)+(\Delta x)^2,$$

and if Δx is small the error is approximately $2x(\Delta x)$. And more

generally, the error in $f(x)$ due to taking $x+\Delta x$ for x is, under the proper conditions on $f(x)$ and its derivatives, $\Delta x \,.\, f'(x)$ approximately.

If I may illustrate my argument by referring to one point in the Elementary Calculus to which I have already referred,* without giving rise to the idea that this is in any sense an attack on the book, it seems to me only to confuse the beginner if, having obtained

$$\Delta S = 2x(\Delta x) \quad \text{approximately}$$

we go on to say

$$dS = 2x \,.\, dx \quad \text{accurately,}$$

unless we are trying, as the authors there are, to make clear the distinction between the increment ΔS which we use, and the differential dS, which the pupil need never hear of till much later. It is, to my mind, no justification to add "it is easy to write down the approximate relation $\Delta S = 2x(\Delta x)$ by thinking of the accurate formula $dS = 2x \,.\, dx$".

I hope I have made my main trouble clear. The early teaching of *errors* requires very careful treatment if we are to avoid the all-too-prevalent delusion that dy/dx is an approximation. I am afraid that if we add differentials to our deltas we shall start the delusion that Δy is exactly $\dfrac{dy}{dx}\Delta x$. We have already one exact thing—the derivative—and one approximate thing—$\dfrac{dy}{dx}\Delta x$; they should be enough to prevent confusion. I am sure that the calculus syllabus in most schools is sufficiently loaded already; I want to make it clear that there are university teachers who will not protest at the appearance of entrants who have never heard of differentials.

I am glad to see that one of the authors of the book to which I have referred is here this afternoon. I hope that he will give us his side of the question and that, as far as possible, the whole discussion will be confined to practical lines.

Mr. C. O. Tuckey (Charterhouse) said he was one of those who welcomed the book in question, not because he thought it was more accurate or better from the point of view of abstract mathematical theory, but because it made it unnecessary for teachers to badger their pupils with distinctions which, though they admitted them verbally, they never really got into their heads. If one wanted to get rid of the idea that $\dfrac{dy}{dx}$ was $dy \div dx$, there was only one method to be adopted with the average stupid boy. This was to cease to write it as a fraction. If one wrote it $\dfrac{dy}{dx}$ and presented it to the type of boy who took lower mathematics for Woolwich, although the boy might admit that it was not a fraction he really believed it was a fraction, and his own experience of some twenty years was that nothing would convince the boy that it was not a

* Durell and Robson, *Elementary Calculus*, I. (1933), p. 30.

fraction. The people who did not want to teach differentials at the elementary stage should have invented some different notation; for instance, $D_x x^2$, which would rigorously exclude $\frac{dy}{dx}$. If one introduced differentials one had $\frac{dy}{dx}$ as a fraction. There was no objection to trying to prove that a derivative was accurate; he was all in favour of trying to prove that, but $\frac{dy}{dx}$ was allowed to mean $dy \div dx$.

The second point where the introduction of differentials came in was this kind of thing. One wrote down the integral of $\int(1+x)^3\,dx$ and wanted that to be equal to the integral of $\int(1+x)^3 d(1+x)$. Either one had to justify that by an argument, which the boys invariably forgot and nearly all of them regarded as "fudge", or one had merely to say that the differential dx was in fact equal to the differential of $1+x$.

He thought the improvement he had mentioned would save about five hours' work, and that the net result of introducing differentials early in the course was to save some ten hours' work; a considerable advantage.

Mr. A. Robson said he thought Professor Temple would agree that the three methods he had spoken about would all be unsuitable for teaching to children of the age of fourteen. It was children of the age of fourteen, fifteen or sixteen that he and his co-author had in mind when they wrote the book to which reference had been made. That book was not intended to be a particularly formal or rigorous presentation of the subject. It was meant to be accurate, but it was only intended to be an intuitive course, with graphs and so forth. The question seemed to him really to be whether it was worth while, in that very elementary teaching, to be in the position that Mr. Tuckey had spoken of, the position of being able to say that $\frac{dy}{dx}$ was equal to $dy \div dx$. He thought Mr. Tuckey's second equation did require more justification than saying the differential of x was equal to the differential of $1+x$.

Suppose one had $y=f(u)$, where u was a function of x; this was a place in which the notation of differentials was extremely useful. Why could one say that $dy=f'(u)du$, $du=g'(x)dx$?

Mr. Phillips, who had written the article in the *Gazette*, was advocating the introduction of differentials chiefly from the point of view of the functions of two variables, but he (the speaker) thought it was worth while introducing them for functions of one variable.

What he would like to learn was whether it was generally considered worth while to be able to do the kind of thing to which he had referred.

Dr. W. F. Sheppard urged that the important thing for the novice is that he should become familiar with the idea of associated varia-

tions—the idea that if u is a function of v then v is a function of u. When he has realised this, he can go on to general formulae.

Mr. Robson said that he had meant to emphasise that dx is an arbitrarily chosen number that can be anything except zero, and that dy is then definitely defined in terms of dx. The object was not to make it clear that dx and dy were exactly on the same footing, but that they were not.

The President asked whether, if boys were taught differentials, there was not a serious risk of their running away with the idea that the whole of mathematics was only approximately, and not quite, true.

Mr. Robson replied that that was only the case if dy and dx were confused with δy and δx. Some trouble would have to be taken to see that the boys did not confuse them.

Mr. G. W. Ward said that the work with differentials was essentially work with linear forms. He started with the definition, "y is a differentiable function of x if Δy can be expressed in the form $A\Delta x + \epsilon\Delta x$, where A is independent of Δx and ϵ tends to 0 as Δx tends to 0". As a sort of act of grace we may write dx for Δx and define dy as being Adx. From these equations we obtain $\frac{dy}{dx} = A$ and $A = \lim \frac{f(x+\Delta x) - f(x)}{\Delta x}$.

The use of linear forms was illustrated by a proof of the formula $\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz}$. Taking y as a differentiable function of x and x as a differentiable function of z, we have the equations, $\Delta y = A\Delta x + \epsilon\Delta x$, $\Delta x = B\Delta z + \epsilon'\Delta z$, whence $\Delta y = AB\Delta z + (A\epsilon' + B\epsilon + \epsilon\epsilon')\Delta z$. The term $A\epsilon' + B\epsilon + \epsilon\epsilon'$ tends to 0 with Δz. It follows that $AB = \frac{dy}{dz}$. But $A = \frac{dy}{dx}$ and $B = \frac{dx}{dz}$, and the theorem is proved.

Mr. D. E. Collier (Southern Secondary School, Portsmouth) said he did not think it was at all necessary to introduce differentials into ordinary school mathematics, and any time or trouble taken in avoiding them was, in his opinion, not wasted. He did not feel inclined to change that opinion as a result of what had been said on the subject that afternoon.

Mr. C. G. Paradine (Battersea Polytechnic) said that he found it necessary to teach differentials in order to help engineering students who met the notation with other lecturers.

He wondered whether the non-equality of the two symbols dy and dx might be overcome by saying that Δx meant any interval including the point, dx and dy quite independently being any intervals whatsoever, measured on the tangent line. Then it seemed to him that the theorem

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

ought to be treated as pure arithmetic. The only difficulty in the theorem was when the limits were not finite. In all elementary

treatments it was assumed that the limits were finite. He did not believe that most teachers proved the limits theorem; he usually made a reference to it, only saying that by a theorem of limits so-and-so happened. It seemed to him that, if dy and dx were taken on the tangent line, there was no objection to treating the theorem as a statement of simple algebra.

Mr. Hope-Jones (Eton) suggested that a cause of the difficulty experienced was that elementary teachers and advanced teachers used the word "differentials" in different senses. The intervals on the tangent seemed to him to provide a possible meaning of the word "differentials" which made it clear to the ordinary boy that the thing proved apparently approximately at first for pieces of arc was quite literally and exactly true for pieces of tangent.

He would plead for greater tolerance in the use of abbreviations. If the subject was to be taught without boring the boys intolerably, it was necessary to make statements which were not themselves true but were abbreviations, and he thought it should be sufficient to stop occasionally and point out at full length what it is that was being abbreviated. The rest of the time one could go ahead with the short statements that were not literally true, so as not to choke interest in the subject-matter by interminable explanations.

Mr. J. T. Combridge, in reply, said he could assure Mr. Hope-Jones that Professor Temple and himself and everyone else meant by "differentials" simply two quantities whose ratio was equal to the derivative of $f(x)$. If anyone liked to use $\frac{dy}{dx}$ and to explain that dy and dx were two quantities defined by the fact that their ratio was equal to their ratio, he was quite welcome to do so!

He had been very pleased to hear Mr. Tuckey's remarks, and if none but people like Mr. Tuckey were going to use the elementary books to which he had referred he would feel much more happy about the future. He could express himself best in the opening words of an alderman of one of our largest seaport towns, who, when called upon to take part in a discussion on the League of Nations, began: "Ladies and Gentlemen, I must confess I am septic"!

Note added 24th January, 1934.

My reply to Mr. Hope-Jones was the truth, but not the whole truth. There are really two kinds of differential, for which different names are certainly desirable though not available. One kind is used to define the derivative; the other is defined in terms of the derivative in accordance with the pair of definitions I quoted. Professor Temple's paper—an admirable summary of the three species of differential of the first kind—was intended to show that only the second kind are teachable in schools. But it was to the second kind that we were all referring—at least, I hope so—in the subsequent discussion. It is also to these that I refer in what follows.

Mr. Tuckey gave two reasons why some teachers welcome the early introduction of differentials :

(i) it enables them to tell pupils that dy/dx is *really* a fraction ;

(ii) it enables them to explain much more shortly why

$$\int (1+x)^3\,dx = \int (1+x)^3\,d(1+x).$$

Mr. Robson gave a third reason :

(iii) it enables them to deal more easily with the differentiation of a function of a function.

My objection to (i) was indicated in my reply. If the "stupid" boy really believes that dy/dx is a fraction after the master has told him it is not, does it raise his opinion of the master to be told instead that it *is* a fraction, and that the two quantities dy and dx, whose ratio it is, are any two quantities whose ratio is equal to it ? The simple truth is that, owing to the historical development of the differential calculus, we sometimes write the derivative $f'(x)$ in the form dy/dx, and that $f'(x)$ may be written as the ratio of any two quantities whose ratio is equal to $f'(x)$. But $\tan x$ may be treated in the same way ; this is not a property of the calculus.

Reasons (ii) and (iii) are intimately connected, and (iii) presents no real difficulty. The rule for differentiating a function of a function may be established quite easily without the use of differentials —see, for example, Mr. Robson's own book (p. 50). It remains only to show that (ii) can be reduced to (iii).

It has already been observed that it is unfortunate that we have no such notation as $f^{-1}(x)$ for $\int f(x)\,dx$, since the latter is a function whose derivative with respect to x is $f(x)$—*i.e.* if

$$y = \int f(x)dx,\ dy/dx = f(x).$$

Suppose we wish to evaluate

$$\int x(1+x^2)^{\frac{1}{2}}\,dx.$$

Denote the integral by y, so that $\dfrac{dy}{dx} = x(1+x^2)^{\frac{1}{2}}$.

Put $1+x^2 = u$, so that $\dfrac{du}{dx} = 2x$.

Then $$\frac{dy}{du} = \frac{dy}{dx} \Big/ \frac{du}{dx} = \tfrac{1}{2}u^{\frac{1}{2}},$$

and therefore $$y = \tfrac{1}{3}u^{\frac{3}{2}} + \text{const.} = \tfrac{1}{3}(1+x^2)^{\frac{3}{2}} + \text{const.}$$

Not only does this method avoid having to write $dx = \dfrac{dx}{du}du$, with the consequent separation of the sign of integration from the dx, but it removes an objection which others beside myself have to Mr. Tuckey's way of writing $\int (1+x)^3 dx = \int (1+x)^3 d(1+x)$—or, in the

example I have just used, $\int x(1+x^2)^{\frac{1}{2}}dx = \frac{1}{2}\int (1+x^2)^{\frac{1}{2}}d(1+x^2)$. This objection, based on experience not only with classes but with backward individuals, is that pupils who find difficulty with $\int (1+x)^3 dx$ find still greater difficulty with such forms as $\int (1+x)^3 d(1+x)$ and $\int (1+x^2)^{\frac{1}{2}} d(1+x^2)$. I have always found that the replacement of $1+x$, or $1+x^2$, or whatever it may be, by another variable is a safer and more easily comprehended process, and though it is longer it can often be short-circuited and reduced to a method which is substantially that of Durell and Robson (*Elementary Calculus*, Vol. I, p. 92).

J. T. Combridge.

GLEANINGS FAR AND NEAR.

961. Mathematical Society, Spitalfields.

It is curious to find that a century and a half since, science found a home in Spitalfields, chiefly among the middle and working classes; they met at small taverns in that locality. It appears that a Mathematical Society, which also cultivated electricity, was established in 1717, and met at the Monmouth's Head in Monmouth-street, until 1725, when they removed to the White Horse Tavern, in Wheeler-street; from thence, in 1735, to Ben Jonson's Head in Pelham-street; and next to Crispin-street, Spitalfields. The members were chiefly tradesmen and artisans; among those of higher rank were Canton, Dollond, Thomas Simpson, and Crossley. The Society lent their instruments (air-pumps, reflecting telescopes, reflecting microscopes, electrical machines, surveying instruments, etc.) with books for the use of them, on the borrowers giving a note of hand for the value thereof. The number of members was not to exceed the square of seven, except such as were abroad or in the country; but this was increased to the squares of eight and nine. The members met on Saturday evenings: each present was to employ himself in some mathematical exercise, or forfeit one penny; and if he refused to answer a question asked by another in mathematics, he was to forfeit twopence. The Society long cherished a taste for exact science among the residents in the neighbourhood of Spitalfields, and accumulated a library of nearly 3000 volumes; but in 1845, when on the point of dissolution, the few remaining members made over their books, records, and memorials to the Royal Astronomical Society, of which these members were elected Fellows. This amalgamation was chiefly negotiated by Captain, afterwards Admiral Smyth.—John Timbs, *Clubs and Club Life in London*, p. 403.

[John Timbs (1801-1875): according to the *Dictionary of National Biography* "his works run to over 150 volumes: compilations of interesting facts gathered from every conceivable quarter. In recognition he was elected F.S.A.".

For another account of the Mathematical Society, see De Morgan, *Budget of Paradoxes*, i, 376-383 (1915).]

962. By Threes. A musical composition, a mathematical proof, needs for its greatest esthetic effect a sense of suspense during its development, with the full resolution deferred until the end. If all good things come by threes, we might add the detective story as the third example of this type.—E. S. Allen, in a notice of Speiser's *Die mathematische Denkweise*, *Bull. Amer. Math. Soc.*, vol. 39, p. 484. [Per Prof. E. H. Neville.]

MATHEMATICS IN CENTRAL SCHOOLS.*

Mr. G. T. Clark (Mathematics Master, Loughborough Central School (Mixed)), who opened the discussion, said: A President of this Association once announced for his Presidential Address a subject which, I think, he would be the first to admit was likely to sound enticing even to non-mathematicians, but which in effect was certainly only of interest to those who had made mathematics at least part of their life's work. I propose to formulate the converse of his example in that, having been announced to read a short paper on a subject apparently of interest only to teachers of mathematics, I intend to deal with it in a manner which may possibly appeal more to non-mathematicians. I do this because I find the object of Central Schools and the work carried out in them are not well understood by teachers working in other types of schools, and surely before we can discuss mathematics in Central Schools it is essential to understand something of what they do and what they are. My remarks must be understood to be limited to London Central Schools in general and my own in particular, and, as a servant of the London County Council, I have to add that that body cannot accept any responsibility for opinions expressed in this paper.

Central Schools are sometimes referred to as a type of Secondary School, and a speaker at a London Branch Meeting suggested that the only difference he could observe between the two types of schools was in the financial remuneration of the staff. This gentleman certainly found one important difference, but that affects the staff and not the school. Let me say at once that the Central School is not intended to rival the Secondary School or to function in its capacity; both have clearly defined work to accomplish and as long as this is realised one cannot be mistaken for the other.

The object of the Central School is to provide further education for boys and girls up to the age of 15+, and when that has been said any reasonable subject which provides for this may be legitimately included in a Central School curriculum, which usually has a bias towards the main occupation to which boys and girls in the neighbourhood are likely to proceed. As the London County Council treats selective Central Schools far more generously than ordinary Elementary Schools in the matter of financial allowance, it is naturally expected that full benefit should be taken of this, and, it being considered that this would not be obtained in a school course of less than four years, all parents are required to give an undertaking that, if their child is admitted to a Central School, he will be retained there until the completion of such a course. As a matter of fact, the benefits of a Central School education are becoming so well known that there is an increasing desire on the part of many parents to keep their children at school for a fifth year, that is, until they attain the age of 16+. At the Loughborough Central School we have a class of twenty-four boys and girls, all over 16 years of age,

* A discussion at the Annual Meeting, 5th January, 1934.

and another in which practically all are just under 16. No fees are payable in Central Schools, and indeed in Board of Education statistics such schools are probably referred to as Selective Elementary Schools. All text-books and stationery are provided free. In certain cases, depending on the income of the parents, monetary grants are made in order to assist pupils to remain at school after the legal age of exemption from attendance and thus complete their Central School career. When they leave school, pupils also have the privilege of attending Polytechnics and Senior Technical or Commercial Institutes at a much reduced fee, if they have completed their Central School course. Every endeavour is made by the Heads and their staffs to find employment for their pupils, and a Central Schools' Employment Committee of the Ministry of Labour exists, which does excellent work in finding situations for boys and girls from our schools.

How are children selected for Central Schools? Every half year a compulsory scholarship examination is held throughout London. As a result of this, certain boys and girls are awarded scholarships to Secondary Schools, where they have the opportunity of continuing their education until they reach the age of 18. Here we see an important difference between Secondary and Central Schools, where, as I have previously said, very few pupils are retained after 16+. The remainder of the candidates for the scholarship examination are classed A, B, C, etc., according to the marks received. The Head of the Central School now takes a part in the proceedings, as it is his duty to visit all his contributory schools to interview the Heads of these schools in respect of pupils who wish to come to his school and who must usually be among those classified by the scholarship examination. Now it will be observed that he has at his command a possibility of obtaining a group of pupils whose educational attainments are approximately the same; this, it must be confessed, is where the Central School scores over the Secondary School. It is true that the latter secures the brilliant scholarship child, but I understand there are others admitted who cannot be described by either of these adjectives. Some Heads of Central Schools are able practically to limit their selection of pupils to those in Class A, others include Class B and others may include Class C. The inclusion of any particular class depends on many factors, which would probably not be of much interest to you, but I must point out that the advantage of the narrowness of the educational attainment band, if it may be so described, is lost if the Head selects pupils far down in the classification. He is, of course, able to grade his incoming pupils should he so desire, and I feel pretty confident that, if he takes eighty, which is the usual number taken, he should get one quite good class and another which may be described as fair average. Unfortunately, this second class is often labelled 1B, a nomenclature I dislike, because so often an inferiority stigma is thus imparted which remains with the class throughout the whole of the school. That difficulty is no doubt familiar to you.

The staff of Central Schools consists of persons well qualified in the one or two subjects which they have to teach, and it seems now to be almost a necessity to have an honours degree in order to be eligible for a post in these schools. Some curious combinations of subjects taught by individual teachers may be found. For example, one of my colleagues combines mathematics with typewriting, another combines mathematics with shorthand, another mathematics with art, but I can conceive this may happen in other types of schools. Such combinations of subjects produce a complete change of outlook and tend to relieve any monotony of always teaching the same or allied subjects.

What subjects are taught in Central Schools ? In the first two years of our course we teach the usual subjects that may be found in any Secondary School, with the exception of Latin or Greek ; the modern foreign language taught is practically always French. The course up to this point is perfectly general in its application. From the third year onwards the course receives a vocational bias, sometimes commercial, sometimes industrial, and sometimes both. This bias is often determined by the neighbourhood in which the school is situated and the posts to which the pupils are likely to be appointed.

The school in which I work has a commercial bias, so that in the third year we begin to teach shorthand and book-keeping, to which is added typewriting in the fourth year. The Head has practically full power to select subjects and fix the time devoted to each. I think that this is the place to stress a great point for the Central Schools ; they are entirely free to construct any syllabus in any subject, a privilege that should be appreciated and guarded very jealously by the staffs of these schools. Such a condition of things makes a paper on my subject rather difficult, because, though we do certain things in mathematics at the Loughborough Central School, we may be isolated in that respect, as the other eighty Central Schools in London may be doing something quite different.

Before I proceed with the subject of mathematics, I should like to say a word about examinations. Some Heads do not look with favour on examinations, because they consider that taking them limits the field of their syllabuses, although surely it must be admitted that examination syllabuses are usually constructed with certain aims in view and on a general as well as a generous plan. On the other hand, another Head may consider that examinations are good both for teacher and for taught and, at any rate, give the successful candidate a certificate with which to go out into the world, and many employers demand certificates when engaging applicants for situations. Again, in some districts in which Central Schools are situated it is difficult for parents to provide the necessary examination fees, and consequently candidates for examinations are not forthcoming. On this account some Heads are nervous that, on a return of examination results, their schools are likely to show up badly for reasons outside their control, and so they are naturally prejudiced against them. I may add that the London

County Council will not pay the fees of any pupil taking examinations from Central Schools. This is rather a strong indication of the official attitude towards sending in pupils for examinations, but in any case, for reasons of economy, it is hardly probable that they would do so. The Loughborough Central School is situated in a middle-class district where, although money is not too plentiful, parents are usually willing to pay the fees required for the examinations we take. We send in candidates for the single subject examinations of the Royal Society of Arts and the Chamber of Commerce, which are taken as the scholar proceeds through the school. Those who are capable, and who stop for a fifth year, are entered for the Oxford School Certificate Examination. Other schools with a commercial bias often adopt this procedure. Pupils are also entered for Civil Service and County Council Examinations and also for the examinations of the Royal Arsenal and the Air Force.

Now I come to the real subject of the paper, mathematics. As I have previously said, there are wonderful possibilities of much variation in this subject. Some Heads consider that mathematics should occupy a very small place in a commercial curriculum, the time saved being given perhaps to studying a second foreign language or to studying some aspect of a commercial education which would otherwise be crowded out. I think, however, that you would certainly find that in all Central Schools arithmetic, algebra, and geometry are being taught, while in some you would find in addition trigonometry and the elements of calculus. The syllabuses and methods employed would, of course, be numerous. A few months ago, as a member of the Boys' Schools' Committee of the Association, I was invited with others to prepare a questionnaire with respect to mathematics in Central Schools. Although the questionnaire was actually prepared, it was not sent out, for reasons which need not be enumerated here. I feel quite confident that, had it been submitted, some very illuminating replies would have been obtained which would have been very difficult to collate.

Let me now particularise with regard to my own school, where the Head delegates to each senior assistant teaching a particular subject the task of preparing a syllabus in that subject for the whole school.

Arithmetic naturally receives prominence in a school with a commercial bias. We like to lay stress on the importance of correct computation in working out examples and on the importance of facility in the use of decimals and decimalisation. The use of contracted multiplication and division probably receives more attention than in other types of schools, because their use is essential in the commercial examinations that we take. As a whole, I should say we devote more time to the correct working of mechanical types of sums than to those which require clear thinking for a correct solution. To this end we have frequent tests in what we call mechanical arithmetic, in which the child has to work twenty straightforward sums in an hour. Such a test requires speed, accuracy, and a sound knowledge of method. In working sums in profit and loss

we as often estimate gain per cent. on the selling price as we estimate it on the cost price. Our syllabus is intended to be sufficiently wide in its outlook to include processes likely to be met with in the early business life of a boy or girl. Half-way through the fifth year course a pupil is expected to be able to work examples of a standard suitable to a School Certificate candidate, and at the end of the year those who still remain will have had a course involving annuities, insurance, harder shares and stocks, calculation of interest in hire system purchases, etc. At the end of the fifth year a pupil should be able to take the Stage 3 Examination of the Royal Society of Arts, one of the most difficult examinations in this subject with which I am familiar. I must confess, however, that towards the end of the fifth year the boys and girls still remaining at the school have become unsettled, owing to their very laudable desire to obtain posts, so that the actual taking of the examination is a very rare occurrence. Many scholars, however, take the Stage 2 Examination in their third or fourth year.

With regard to algebra, our syllabus in this subject probably differs very little from that of the ordinary School Certificate course in a Secondary School. It includes simple, simultaneous, quadratic and simultaneous quadratic equations, with problems based on them, graphs, theory of indices, logarithms, and arithmetical and geometrical series. I have not, of course, in this brief list of types exhausted the resources of our syllabus, but my intention is simply to give you some idea of the ground covered. As we occasionally present candidates for the Junior School Commercial Examination of the Royal Society of Arts, some of the examples we work are assumed to have commercial colour about them. We usually attack the subject from the problem point of view, but next year we are thinking of attacking it through the formula and symbolic arithmetic method.

The first six months of the course in geometry is devoted to teaching the use of instruments and in carrying out simple instructions with them. In this part of the course we also work simple problems involving intersection of loci, scale drawing, measurement of angles, knowledge of compass directions, etc. We find that the girls—we are a mixed school—take much longer to learn how to use the instruments than the boys, who may probably have had previous acquaintance with them, but at the end of the six months' course there is very little difference in the capabilities of either sex, except for a few girls who do not seem to be able to make much headway. The next six months are given to an endeavour to learn the important facts of elementary geometry by practical experiment. At the end of this period a child is expected to know something of the properties of the various kinds of angles, parallels, congruence, etc., and to be able to work simple problems based on these ideas. We have frequent little talks on the history of the subject and the derivation of some of the words we have to use. We try to get the children to write out little proofs in a geometrical style from the commencement of the course. At the beginning of the second year we

commence theoretical geometry. This consists of the usual matter which one would expect to find in such a course and includes similarity and some solid geometry. We introduce the geometry of the circle wherever it is possible to do so, and frequently break the text-book order for this purpose. It is in this subject that I find my greatest difficulty in teaching mathematics to Central School pupils, for, although they are able to work out numerical problems in the subject with fair success, it is only the few who are able to tackle successfully any but simple riders, until they arrive at their fifth year, when there is definitely some improvement, probably because more time is given to the subject.

A short course of numerical trigonometry is taken towards the end of the fifth year course.

The time allowed for mathematics varies from three hours to four and a half hours per week, but, as these times may have to be broken into for swimming or choir, it will be realised that our time allowance is not too much.

I feel that I have had to cut out much that I should like to have said on Central Schools and also that I have not given enough consideration to the real subject of the paper, but at the same time I hope that I have said sufficient to provoke a discussion and elicit questions. I suggest that perhaps some of the following points may form a basis for discussion :—

(1) Should mathematics of a School Certificate standard be taught to boys and girls in a Central School ? In this connection a writer has recently given as his opinion that it ought to be advantageous to a child in a Central School to dispense with a fair proportion of the arithmetic, reduce algebra to simple symbolic arithmetic, and confine geometry to results and facts obtained by experimental methods. This would certainly give an opportunity for studying other subjects, but to what subjects would preference be given ?

(2) Is it conceivable that a practical course in geometry would meet the needs of a Central School better than a theoretical course ?

(3) Is it likely that a uniform scheme of mathematical teaching throughout Central Schools would be acceptable to them ?

(4) Should trigonometry always find a place in the mathematics course, even to the exclusion of some of the usual mathematics taught ?

(5) Would it be worth while for the Mathematical Association to form a Central School Committee to advise upon and consider schemes of study in mathematics in this type of school ?

In conclusion, it is obvious that if one examines the list of members of the Mathematical Association very few Central School teachers can be found. I think one reason for this is that an idea is prevalent that the Mathematical Association consists of mathematical highbrows who are interested only in the higher branches of the subject and who are not in the least interested in the elementary parts. I am very glad to be able to say that I have found in the Association many ladies and gentlemen who exhibit just as much sympathy,

thought, and help for the teacher of elementary mathematics as for the teacher of more advanced mathematics, and I am certain that, when Central School people realise this, they will come forward to give the results of experiments they have made and be willing to assist in offering their services for any experiments, suitable for Central Schools, which may be suggested to them. I am authorised to tell you that my headmaster will always be pleased to welcome any of you who care to visit the Loughborough Central School at any time convenient to you.

Mr. A. W. Riley (Boys' Central School, Stroud): My task is to deal with conditions and practice in Central Schools in the provinces. It is practically an impossible task, as I know of no official source of recent information. Owing to differences of nomenclature, it is not even possible to determine the number of Selective Central Schools in existence, though I believe there are between two and three hundred in the provinces and eighty odd in London. Most of what I have to say relates to the schools of which I have personal knowledge. In any case, for reasons which will appear, there is no such thing as a typical Central School.

My friend Mr. Clark has outlined the general scope of the Central School, and has dealt at some length with the selection of entrants. In these matters, as in the general question of a bias in the course, the same conditions hold good in the schools I know. Mr. Clark has dealt particularly with the London Central School with a commercial bias; so far as my knowledge goes, similar conditions apply also in Provincial Schools with the same bias. I propose to confine myself to the subject of the Boys' Central School with an industrial bias.

Of the industrial bias, we find in the Hadow Report on the *Education of the Adolescent* the following:

"The degree of the bias which is given to the course in boys' Central Schools varies greatly in different areas. In a few districts containing large engineering works, the bent in the direction of engineering is very noticeable, and except for the fact that Central Schools in such districts do not employ trade instructors, their curricula in the last years of the course often bear a strong resemblance to those of some Junior Technical Schools. In the Central Schools of a few towns on the sea-board, such as Scarborough and Lowestoft, specialised instruction is provided for boys going to sea. Similarly, some of the Central Schools and classes giving advanced instruction in Devonport, Plymouth, and Portsmouth frame their curricula to suit examinations which determine the admission of boys to the Royal Dockyards. Another type of Central School, which is not uncommon in the north of England, provides courses which serve as the first portion of a curriculum intended to be completed in the evening schools."

In all these types, the mathematics will be strongly influenced by the bias of the school.

In the last paragraph of the same chapter of the Hadow Report, there is an important note as follows:

"The bias should be introduced only after careful consideration of local economic conditions and upon the advice of persons concerned with the local industries. It should not be of so marked a character as to prejudice the general education of the pupils. Adequate provision should be made for the needs of such pupils as may gain greater advantage by following a more general course of study."

The last two sentences indicate one of the differences between the Selective Central School and the Junior Technical School. In the Junior Technical School, it is presupposed that entrants have decided upon their future employment, and they accordingly follow definite trade courses. The bias in a Central School is industrial, but not technical. In the Central School the utility standpoint is subordinated to the cultural, but it is borne in mind that the Central School boy will very likely go into industry at the age of 15 plus, and so his course is planned in order that it may be concerned with topics likely to prove useful later on, but not with specific requirements of any one trade or craft. The second proviso quoted—that provision should be made for pupils who may follow a more general course of study—also prevents the course from becoming too definitely vocational. Unless the school is very big, it will not usually be possible to group such pupils in separate classes, and, in any case, the proper place for them is the Secondary School. I know that many Central Schools run a School Certificate form. I cannot help feeling that this practice is quite wrong; it attempts to do something which can be done better in the orthodox Secondary School, and it tends to stifle the development of the Central School along its own lines.

A course with an industrial bias, in particular a course in mathematics, must be planned with these points in view. First, the great majority of the boys will cease to study mathematics as a cultural subject after the four years' course in the Central School. Second, the mathematics learnt should help the boy along in his later work in evening schools as far as possible—in my experience most ex-Central School boys attend evening schools where they are available. The course must be planned with regard to the industrial needs of the district. Third, the course must be such that it can be used, if necessary, as the basis of a more advanced general course—as for School Certificate—in individual cases.

As local needs are to determine largely the nature of the industrial bias in any particular school, it is clear that anything like a common syllabus is out of the question. Teachers of mathematics, as of other subjects, are working out their own particular salvation in the two hundred odd Central Schools up and down the country. At present they are untrammelled by external requirements as regards examinations, and it is to be hoped that they may long remain so. I have found that the industrial needs often make themselves felt through the type of course required for evening school students—as for example the National Certificate Courses in Engineering and Building. Some years ago this Association published a report on *The Teaching of Mathematics in Evening Technical*

Schools. In that report emphasis was laid upon the fundamental position of Algebra in any scheme of technical training. I think the work of an industrially biased Central School has an important bearing on this point. After teaching and examining in evening technical schools, I have found that the *average* elementary school leaver at 14 plus, who attempts to commence Algebra in evening school, cannot reach real proficiency by the age of 16 plus, at which time he is supposed to embark on a Senior Course leading to the National Certificate. If, however, he—and I speak of the average boy—can pursue in a day school a course of Algebra for four years from 11 plus, he has a reasonable or a good chance of reaching the necessary standard for his Senior Course in evening school—and that without framing an *ad hoc* syllabus, which is important, as I have already said that it is undesirable for a Central School to be tied down to a syllabus for an external examination.

I now venture to offer some suggestions for a syllabus worked out on the lines I have laid down. This syllabus must not be taken as representing anything more than my own opinions at the moment. As a whole it is not even in force in my school at present, though all parts of it have been done by one group of boys or another. I claim no finality for it; I am tolerably certain that next year my work will be influenced to some extent by the paper Professor Hamley has just given.

In Arithmetic the underlying idea is that the boy shall be able to handle number confidently and accurately, so that to think in terms of numerical relations becomes a part of his nature. He must also realise that all measurement is approximate. The matter of approximation I have found difficult to teach, yet it seems to be vital in practical mathematics. The topics "approximation," "degree of accuracy," "significant figures" occupy little space on the syllabus or in the text-book, but they are ideas which have to be kept in the foreground of consciousness of the boys all through the course. There is plenty of scope for practical work here, especially in a school with an industrial bias, where much time is spent in the workshops. Verniers, micrometers and limit gauges may become much more than text-book topics. Certain arithmetic lessons are spent in the physics laboratory, in default of a mathematics laboratory; the sections on measurement, density and so on which usually occur in text-books of elementary science are dealt with in the Arithmetic lesson.

The content of the Arithmetic syllabus is fairly orthodox. In the higher forms considerable time is devoted to mensuration, the Arithmetic lesson often becoming indistinguishable from the Geometry lesson. Commercial arithmetic is, of course, rather ruthlessly sacrificed, though for the sake of utility and general knowledge brief notice must be made of it.

In Algebra, we are less orthodox. It is quite certain that very few boys at the end of the four years' course could tackle successfully a School Certificate paper in Algebra—or, for that matter, in Geometry either. The principal idea of the course is functional

relationship, and the graph occupies an outstanding place in the syllabus. Theoretically, I should like to be able to approach Algebra through the formula, but I find that the rate of progress in Arithmetic is too slow for Algebra to wait upon the occurrence of new formulae, that is, if the Algebra is to be more than an occasional side show for Arithmetic. So we make a concurrent approach through the problem and through the graph. The problem approach does not involve any special treatment. The graph is tackled as early as possible, generally to meet the demands of the geography master in connection with temperatures and rainfall. Once begun, progress is rapid and fairly easy, as the boys like drawing and using graphs. The idea of a variable or an unknown comes naturally when they can see them, and watch the variable varying. I find that the earlier one starts the graph the more readily is directed number assimilated. Again, the plotting of experimental data is most helpful in illustrating the approximate nature of measurement, when once the boys come to regard the graph as greater than its points. For early work, especially with statistical graphs, I should like to draw the attention of any who may not have seen them to the set of about twenty graphs of commercial statistics which is published every week in the *Manchester Guardian Commercial*. Having begun graphs rather earlier than usual and made them the backbone of the Algebra, we reach in the fourth year the deduction of linear laws and some other simple cases from sets of experimental data, the data where possible being obtained in the laboratory by the boys themselves—as for example for Hooke's Law and the Pendulum Law. With a good fourth year class, the graphical work has been used as an introduction to calculus, but this is exceptional. The course in ordinary Algebra proceeds in conjunction with the work on graphs, but is reduced to a minimum. Multiplication and division of expressions of more than three terms are barely touched; "Harder Examples" in simultaneous equations, factorisation, and fractions are omitted, along with all theoretical work on logarithms, and square root, long method of H.C.F., remainder theorem, and—rather reluctantly—progressions. Work on formulae occupies a very important place, and ratio and variation are treated as fully as possible, with the help of graphical work.

Trigonometry is introduced early in the third year, through the graph $y=mx$. A graph showing values of "m" is constructed and used to solve problems before the terminology of trigonometry, or tables, are introduced. I always feel that the discovery of this entirely new mathematical weapon is the most thrilling moment of the boy's mathematical career. I think the opportunity to handle just one new weapon like this is sufficient justification for the inclusion of trigonometry in the curriculum. Also, the subject is useful practically, it is easy, and it gives the backward boy a new chance to start level with his fellows once more. Our work in trigonometry is numerical and graphical only, and usually goes as far as the sine and cosine rules.

Geometry may provide more scope for originality than Algebra. Here it must be remembered that the boys of an industrial bias will largely enter occupations where some knowledge of practical geometry willl be useful, if not essential. There are two principal requirements: (1) that a boy may be able to read a drawing; (2) that he may be able rapidly to produce a dimensioned sketch of a piece of machinery or furniture. To acquire the necessary facility there is one faculty above all others which needs developing—what I may call "seeing solid." It follows that, whatever we may leave out, Geometry in our schools *must* include some study of spatial relations in three dimensions. The solid geometry is not a matter purely for the mathematics room; it figures in the work of the Art Room and the Workshops. In practice, the mathematics master, the art master and the workshop instructors each give part of the instruction in Geometry, but it is the mathematics master who draws together into a coherent whole the separate threads. The process is apt to prove a little disjointed; from the standpoint of mathematics alone it might be easier for the mathematics master to take the whole course, but parts of the subject are so closely bound up with art and handicrafts that it seems artificial to exile the Geometry from its natural surroundings.

For the first few months the work in the mathematics room is probably identical with that in most Secondary Schools, being directed towards a mastery of drawing technique. When the necessary facility has been gained, it is used in the art room and the workshops. Meanwhile, the two latter departments concentrate on the freehand representation of form, using a kind of conventionalised perspective drawing, in the one case to produce a picture pure and simple, in the other case to produce a dimensioned sketch from which to construct a model in wood or metal. During the preliminary period the mathematics master covers experimentally some elementary facts and constructions, including first ideas of the circle, which is introduced as a locus as early as possible. Parallel to and in conjunction with graphs throughout the school, the study of loci is developed, a good deal more time being given to plotting loci than I believe is usual in Secondary Schools. The work on loci is very useful in the study of mechanisms in the science course.

As soon as the facility in drawing is acquired, the art master probably uses the knowledge of the regular polygons to help his work in design. The workshops introduce orthographic projection. On the other hand, the mathematics class proceeds to develop from a mathematical standpoint ideas of form already gained—the idea of symmetry is an instance. Work on the geometrical solids follows, from which the boys take back to the metalwork shop their knowledge of the nets of the solids.

This distribution of Geometry is carried on throughout the course, but the more advanced work tends to be concentrated in the mathematics proper. In this way, in the art room geometrical design is carried forward to its applications to architectural details. Perspective is also dealt with in the art room. In the workshops

the geometrical work keeps pace with practical needs. As suitable opportunities occur, oblique and isometric projections are introduced. When work on sections of solids, or on intersections of surfaces, tends to absorb too much time which ought to be spent at the bench, the Geometry migrates to the mathematics room. The teachers of science and geography similarly make use of the mathematics.

The advantage of the method here set forth is that the boys really do see that mathematics is not entirely an academic subject. The chief disadvantage is that the unfortunate mathematics master is at the beck and call of everybody else.

In the intervals between attending to other people's requirements, the mathematics master pursues a more normal course of Geometry, mainly on practical lines. Deductive Geometry is done, but does not occupy a prominent place, not because it is considered unimportant, but because the practical geometry is considered more important. Sufficient is done to show the unity of the whole geometrical framework, care being taken to point out that the whole field may be covered by the deductive method. One has to remember, too, the occasional pupil who will proceed to a more academic course.

In the practical geometry, emphasis is laid on those topics which are of special use for the applied geometry already dealt with. Loci, the circle, the construction of scales, the area of a trapezium, similarity, are all important. Congruence and parallelograms are less prominent. Most of the area propositions are omitted, Pythagoras' Theorem being demonstrated by one of the methods of dissection. The practical plane geometry is not principally an end in itself, but a means to the more important end of solid geometry.

The difficulties of carrying out such a scheme of work are many; time does not permit my particularising about them. The chief difficulty is probably that of staff personnel. Work in a Central School demands a good academic background, originality and initiative in teaching, and unflagging energy. It makes a very heavy drain on the teacher's resources, but is intensely interesting.

The other day Dr. George Dyson, in his Presidential Address to the Conference of Educational Associations, made a strong criticism of the present educational system. I will quote from the press report: "Every school in this land spends most of its time producing clerks. . . . We train the mind, but nine-tenths of this training is by words and ink. . . . If once we could envisage an education which should think more of the actual characters, the actual talents, and the future lives of our present pupils and less of the pens and ink of a specialist past, we should soon begin to march in a new direction. . . . We should send our children into the world with that equipment and aptitude for the culture of their own varied talents and interests which would serve them both in work and leisure for the rest of their lives." While I make no comment on Dr. Dyson's general charge, it seems clear that he does not know of the existence of industrially biased Central Schools. I make no

claim that our Schools have achieved Dr. Dyson's ideal; but I do claim that, at least, we have envisaged it.

Mr. J. Burdon (Headmaster, Downham Central School for Boys) said that some weeks ago he had been informed that the Association was going to discuss the subject of the teaching of mathematics in Central Schools, and, knowing that the numerical strength of teachers in those schools in the Mathematical Association was almost infinitesimal, he felt that they ought to make up in some way in quality for their lack of numbers. The two papers had dealt so adequately with the subject, little was left for him to add.

Mr. Clark and Mr. Riley had dealt with Central Schools and he hoped they had made perfectly clear the type of Central School which was under discussion; he did not know that that had been put quite so clearly as it might have been in the two papers that had been read. One sometimes picked up a copy of the *Times Educational Supplement* and read advertisements asking for applications to be submitted for posts in Central Schools from teachers able to take retarded children. The trouble was, of course, due to the Central School nomenclature. The Central Schools under discussion at the moment were the original Selective Central Schools; he did not know how many of them there were up and down the country. Some educational authorities ran Central Schools which were not selective at all; they were just analogous to the London Senior School. The position of the London Central School particularly, he thought, was somewhat unique. Mr. Clark's Central School was a mixed one with a commercial bias, Mr. Riley's had an almost purely technical bias, and his own Central School was situated in an area where there was no work of any kind near it and the question of bias, which was, of course, rather an important one, was dealt with by having both the technical and the commercial bias, the work on both sides being somewhat on the lines already indicated in the two papers that had been read. It was found that, as far as the boys were concerned, about 50 per cent. at the end of the second year took up the technical bias and the other 50 per cent. the commercial bias. In the Girls' Central School at Downham it was found that a very small number took up the technical bias and a much greater number took up the commercial, which made the running of the school rather difficult, because the same generosity in the matter of staffing shown to Secondary Schools was not extended to Central Schools and, as Mr. Clark had pointed out, teachers sometimes had to be able to take two and occasionally three subjects. Everything which had been said by Mr. Clark and Mr. Riley applied to his own school, which, as he had said, had a dual bias.

On the question of trigonometry, he put that fairly early, it being numerical trigonometry only. He did that because it had been found so necessary in the work in connection with science, and the boys on the technical side in his school proceeded straight to the Senior Technical Institutes of London.

With regard to examinations, the syllabus at his school was not

made to fit any examination at all. In fact, he did not allow any boy to take an examination at any time until he had definitely completed his course; then he could remain for perhaps nine to twelve months, in order to take any particular examination which he required. Some parents liked their boys to have a certificate of some kind. There were a number of boys in Central Schools who ought to be in Secondary Schools, but for some reason or other at an earlier age had not been able to get there, and that type of boy liked to go away from school at the age of fifteen or sixteen with something behind him.

He was prepared to help in any way by answering questions any of the members might care to ask in regard to Central Schools.

Mr. H. S. Newman (Mathematics Master, Barnsbury Central School) said there were some who thought that pupils at Central Schools should not work for the School Certificate examination and there were others who held that they should do so. The aim of the Central School was to provide the advantage of advanced education for those boys and girls who could not go to Secondary Schools, and it was interesting to note the after-careers of some of them. In his own school some of the boys and girls obtained scholarships to Secondary Schools; then one or two went on to Cambridge and others went to training colleges; some had obtained honours degrees. If all those boys had been barred from taking the School Certificate examination at the end of their Central School career, it would have been impossible for them to go on as they had done in after life. He quite agreed that they ought to be in Secondary Schools and not in Central Schools, but the fact remained that they were there, and, that being the case, provision must be made for them. The average ability of the children in Central Schools was very high indeed.

There was also another point of view that should be considered. The education given in Central Schools had a bias either towards commerce or towards industry, but it must not be thought that the boys and girls were going to spend all their future lives at work. The whole trend of modern industry was to make leisure time more and more predominant. Therefore children should be educated for leisure and taught how to spend their leisure in a profitable way. It was impossible for a boy or girl to take up any kind of science, for example, without having a fair knowledge of mathematics to begin with.

Therefore his whole object was to make the scope of mathematics in a Central School as wide as possible, that was to say, he was definitely opposed to suggestions that it should be restricted mainly to arithmetic and to some form of practical geometry.

With regard to the teaching, there was one point he wished to mention. It seems to him that it was most important that the teaching in the first two years should be considered most particularly. If boys and girls at the end of the first two years had a dislike for the subject of mathematics, they would never get over that. Therefore he thought the best teachers of mathematics ought

to be put in charge of the boys and girls in the lower school. If the children in the lower school were trained properly, the work of the higher school was easy.

Miss Hoyle (Spalding High School) said she had been greatly stimulated by Mr. Riley's paper and by his account of the mathematics that he taught to his boys. In her own Secondary School there were many girls who ought to be in a Central School, as they were not really fitted for a Secondary School, and the type of mathematics described by Mr. Riley was the type she would like to teach those girls, in order to give much more practical breadth to the subject. She was, however, in the main syllabus hidebound by examination requirements, and it was only in the intervals that she was able to give the subject a broader view.

Mr. Bushell (Birkenhead) said the thanks of the Association were due to Mr. Clark and Mr. Riley for the papers they had read. A great many members of the Mathematical Association had been for some little time very curious indeed to know what mathematics was being taught in the Central Schools, and they were very grateful to the speakers for telling them something about it.

The fact that mathematics could be taught in Central Schools without the dread of examinations made those schools a very vast and fertile field for experiments. He would like to see an experiment carried out to ascertain whether it was not possible to get rid of the formalities of early mathematics, looking more particularly to geometry and algebra, and trying to extend what might be called the "jam" perhaps in the form of elementary trigonometry. He had been very interested to hear Mr. Riley's statement that the introduction of trigonometry to the boy was a fascinating thing. He thought it might be possible, by cutting out the formalities of early mathematics, to teach elementary trigonometry and a little elementary mechanics as well.

Mr. Riley, referring to Mr. Newman's remarks, said he did not wish it to be understood that he intended by his remarks to condemn the Central Schools which took the School Certificate now. He hoped it would be realised that the fault was one rather of the system which compelled children who ought to be in Secondary Schools to be in Central Schools. As long as those children were found in Central Schools he would be inclined to agree with Mr. Newman and to give them as much as possible of what they needed, but he still maintained that the system ought to allow for the free transfer of boys and girls both ways, from the Secondary Schools to the Central Schools and from the Central Schools to the Secondary Schools. That had been advocated in the Hadow Report, but he did not find that it was carried out in practice.

The President said that, as one of the great majority of people present who knew very little about the working of Central Schools, he wished to express the gratitude of all of them to Mr. Clark and Mr. Riley and subsequent speakers who had given information on the subject. He hoped that mathematical teachers in Central Schools would join the Mathematical Association in greater numbers.

THE ETERNAL TRIANGLE.*

By N. M. Gibbins.

The title of this paper is "The Eternal Triangle"; but titles, like politicians, are sometimes misleading. Only last year, for example, you, Mr. President, skilfully cloaked an address on elliptic arcs under the beguiling title of "The Marquis and the Land Agent: a Tale of the Eighteenth Century". And so now, Ladies and Gentlemen, any hopes you may have entertained of hearing a new solution of an age-old problem must be abandoned. I cannot even say with George Canning:

> "Alas, that partial Science should approve
> The sly Triangle's too licentious self!"

No, my triangle is not sociological but geometrical, though, as you will find to your cost, it is very certainly eternal. But let us leave the shifting sands of metaphor for the security of an ordered exposition.

The object of this paper is to provide a series of revision lessons in geometry, which fall into three groups: those suitable for school certificate or matriculation students, those suitable for post-matriculation students, and those suitable for ourselves. But they are connected by a common thought.

The enquiry was provoked by the request made by a former pupil to solve a well-known rider: "Equilateral triangles are described outwards on the sides of a given triangle. Prove that their centroids form also an equilateral triangle". It occurred to me to try to find out the properties of the figure of the rider and of the following extension of it: "ABC is a given triangle; equilateral triangles BCA_1, CAB_1, ABC_1 are described outwards on the sides, forming a new triangle $A_1B_1C_1$; equilateral triangles $B_1C_1A_2$, $C_1A_1B_2$, $A_1B_1C_2$ are described outwards on the sides of $A_1B_1C_1$, forming another triangle $A_2B_2C_2$; and so on. It is required to find the properties of the figure".

Having done this the next step was to find out whether the investigation could be used for teaching purposes, and the attempt to invite the co-operation of a matriculation form met with surprising results. The boys found something helpful to them for their examination; for they revised the subject-matter of most of their propositions, as well as the usual methods of solving riders. But in addition they had been introduced to the idea of research.

The replacement of equilateral triangles by triangles directly similar to a second triangle leads to a set of properties very like those already found, but with additional features; while the replacement by isosceles triangles of the same base angle leads to a totally different set of properties. But both afford excellent opportunity for the revision of post-matriculation work.

Furthermore, we may describe any triangles whatsoever outwards

* A paper read at the Annual Meeting of the Mathematical Association, 5th January, 1934.

on the sides of a given triangle, and enquire what regularities are to be found. We may also consider what happens when the triangles are all drawn inwards instead of outwards, and find the relations between the two cases. All this will be found to supply an excellent refresher course for the teacher's own mathematical technique !

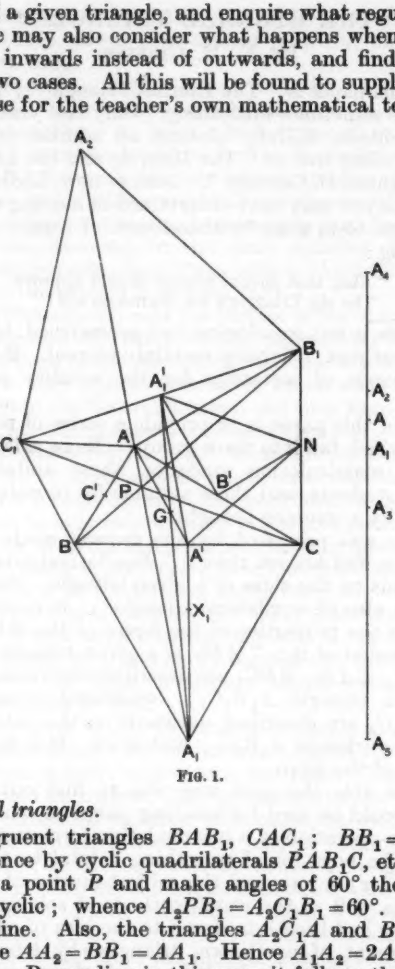

FIG. 1.

A. *Equilateral triangles.*

1. By congruent triangles BAB_1, CAC_1; $BB_1 = CC_1$, = also AA_1; and hence by cyclic quadrilaterals PAB_1C, etc., AA_1, BB_1, CC_1 meet in a point P and make angles of 60° thereat. Again, $A_2B_1PC_1$ is cyclic; whence $A_2PB_1 = A_2C_1B_1 = 60°$. Hence A_2AP is a straight line. Also, the triangles A_2C_1A and B_1C_1B are congruent; hence $AA_2 = BB_1 = AA_1$. Hence $A_1A_2 = 2AA_1$. Similarly for B_2 and C_2. Proceeding in this way, it follows that the vertices of all the triangles $A_rB_rC_r$ lie on three lines which meet at the point P, at which the sides of all the triangles subtend an angle of 120°—the minimum distance point. Further, the spacing of the vertices on their respective lines is in geometrical progression of common ratio 2, as in Fig. 1.

2. Let A', A_1', N be the mid-points of BC, B_1C_1 and B_1C respectively. Then $A'N$ is parallel to BB_1 and equal to half of it ; also

$A_1'N$ is parallel to CC_1 and equal to half of it. Hence $A'N$ and $A_1'N$ are equal and are inclined at an angle of 60°. Hence $A'NA_1'$ is equilateral, so that $A'A_1' = A'N = \frac{1}{2}BB_1 = \frac{1}{2}AA_1$, while $A'A_1'$ is parallel to AA_1. Hence if AA', A_1A_1' meet in G,

$$A_1G : GA_1' = AG : GA' = 2 : 1.$$

Hence $A_1B_1C_1$ and ABC have the same centroid, as have therefore all the triangles $A_rB_rC_r$.

3. Let X_1, Y_1, Z_1 be the centroids of the equilateral triangles. Then $A_1X_1 : X_1A' = 2 : 1 = AG : GA'$. Hence $GX_1 = \frac{1}{3}AA_1$ and is parallel to AA_1. Similarly for GY_1 and GZ_1. Hence $GX_1 = GY_1 = GZ_1$, and they are inclined at angles of 120°. Hence $X_1Y_1Z_1$ is an equilateral triangle, as are all the triangles $X_rY_rZ_r$.

Applying the theorem to the triangle AB_1C_1, it follows that $X_2Y_1Z_1$ is an equilateral triangle, as are also $Y_2Z_1X_1$ and $Z_2X_1Y_1$. Hence X_1 is the mid-point of Y_2Z_2, and so on. Generally X_{r-1} is the mid-point of Y_rZ_r. Thus the points X_r, Y_r, Z_r form a network of equilateral triangles, all having G as their centroid.

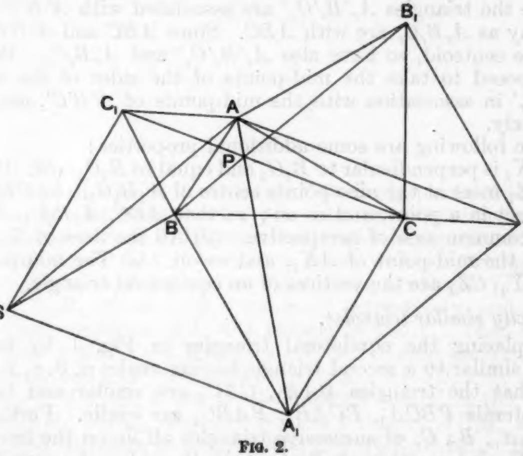

FIG. 2.

4. Draw the equilateral triangles TAA_1, SAA_1 on AA_1 as base. Then by equal and parallel lines, B_1CT and BC_1S are each congruent to ABC, while AB_1T is congruent to SBA_1, and SC_1A to A_1CT. Hence the rhombus $TASA_1$ is equal in area to the figure $AB_1TCA_1BSC_1A$. Hence $2TAA_1 = 3ABC + BCA_1 + CAB_1 + ABC_1$. Again,

$$AA_1B_1 = APB_1 + A_1PB = APB_1 + TPB_1$$
$$= APTB_1$$
$$= ACTB_1 = AB_1CB.$$

Similarly $\qquad AA_1C_1 = AC_1BC.$

Hence $\qquad A_1B_1C_1 = AA_1B_1 + AA_1C_1 + AB_1C_1$

$$= AB_1CB + AC_1BC + AB_1C_1$$
$$= BCB_1C_1 + ABC.$$

Now from § 2, $\qquad BCB_1C_1 = 4A'NA_1' = TAA_1.$

Hence $\qquad A_1B_1C_1 = TAA_1 + ABC,$

or, say, $\Delta_1 = \delta + \Delta$. Since $A_2A_1 = 2AA_1$, the area of the equilateral triangle on A_2A_1 as base is equal to 4δ. Hence $\Delta_2 = 4\delta + \Delta_1$. Similarly

$$\Delta_3 = 4^2\delta + \Delta_2, \ldots, \Delta_n = 4^{n-1}\delta + \Delta_{n-1}.$$

Hence, by addition,

$$\Delta_n = \tfrac{1}{3}(4^n - 1)\delta + \Delta.$$

5. In Fig. 1, $A_1'N = A'N$, $B'N = NC$ and $A_1'NB' = 60° - B'NA' = A'NC$. Hence $A_1'B' = A'C = B'C'$. Similarly $A_1'C' = C'B'$. Hence $A_1'B'C'$ is an equilateral triangle, that is, A_1', B_1', C'_1 are the vertices of equilateral triangles described outwards on the sides of $A'B'C'$.

Hence the triangles $A_r'B_r'C_r'$ are associated with $A'B'C'$ in the same way as $A_rB_rC_r$ are with ABC. Since ABC and $A'B'C'$ have the same centroid, so have also $A_r'B_r'C_r'$ and $A_rB_rC_r$. We may then proceed to take the mid-points of the sides of the triangle $A_r'B_r'C_r'$ in association with the mid-points of $A'B'C'$, and so on indefinitely.

6. The following are some additional properties:

(*a*) AX_1 is perpendicular to B_1C_1 and equal to $B_1C_1/\sqrt{3}$. (*b*) AX_1, BY_1, CZ_1 meet at the nine-points centre of $A_1B_1C_1$. (*c*) CB, B_1C_1, Y_1Z_1 meet in a point, and so on; so that ABC, $A_1B_1C_1$, $X_1Y_1Z_1$ have a common axis of perspective. (*d*) All the lines A_rX_{r+1} pass through the mid-point of AX_1, and so on. (*e*) The mid-points of AX_1, BY_1, CZ_1 are the vertices of an equilateral triangle.

B. *Directly similar triangles.*

7. Replacing the equilateral triangles in Fig. 1 by triangles directly similar to a second triangle having angles α, β, γ, it will be found that the triangles BAB_1, CAC_1 are similar and that the quadrilaterals $PBCA_1$, $PCAB_1$, $PABC_1$ are cyclic. Further, the vertices A_r, B_r, C_r of successive triangles all lie on the lines AA_1, BB_1, CC_1, intersecting at P at which the sides of every triangle subtend angles $\pi-\alpha$, $\pi-\beta$, $\pi-\gamma$; and which is the point for which $A_rP \sin\alpha + B_rP \sin\beta + C_rP \sin\gamma$ is a minimum. By Ptolemy's theorem it follows that $AA_1 \sin\alpha = BB_1 \sin\beta = CC_1 \sin\gamma$; and from this that $A_2A_1 = 2AA_1$, etc. Hence the vertices of the successive triangles are spaced as before on their respective lines. It will be found also that the triangles have in common the centroid of particles proportional to $\sin^2\alpha$, $\sin^2\beta$, $\sin^2\gamma$, placed at their respective vertices. Applying Ptolemy's theorem to the cyclic quadrilateral $PB_rC_rA_{r+1}$, we obtain the result

$$A_{r+1}P \sin\alpha = B_rP \sin\beta + C_rP \sin\gamma;$$

and since $A_rB_rC_r = PB_rC_r + PC_rA_r + PA_rB_r$, it is easy to prove that the area of $A_nB_nC_n$ is of the form $K_1 + K_2 . (-2)^n + K_3 . 4^n$, where the constants depend on the original triangle and the first two to be constructed.

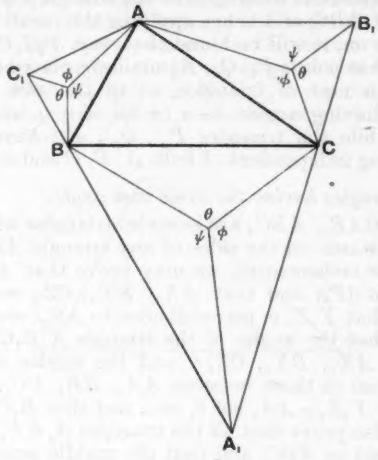

FIG. 3.

8. Let us call a point in a triangle "similarly placed" to another point in a similar triangle when the sides in each triangle subtend the same angles θ, ϕ, ψ thereat, these angles being opposite to corresponding sides. Let P be such a point within a triangle LMN whose angles are α, β, γ; and on NL describe a triangle NLB_1 whose angles at L, B_1, N are $\pi - \theta$, $\pi - \phi$, $\pi - \psi$ respectively, and let P' be the isogonal conjugate of P with respect to LMN. Then it will be found that $\angle MP'N = \angle MLB_1 = \pi - \theta + \alpha$. Hence the sides of the triangle subtend at P' the angles $\pi - \theta + \alpha$, $\pi - \phi + \beta$, $\pi - \psi + \gamma$.

In Fig. 3, if P, Q, R are points similarly placed in the triangles BCA_1, CAB_1, ABC_1 respectively, the triangles AB_1Q, ABR are similar, and therefore so are the triangles AQR, BAB_1. Hence QR makes with BB_1 the angle B_1AQ. Similarly, QP makes with BB_1 the angle B_1CQ. Hence $\angle PQR = QAB_1 + QCB_1 = \phi - \beta$. Thus the angles of the triangle PQR are $\theta - \alpha$, $\phi - \beta$, $\psi - \gamma$. Also if P', Q', R' are the isogonal conjugates of P, Q, R in their respective triangles, the angles of the triangle $P'Q'R'$ are $\pi - \theta$, $\pi - \phi$, $\pi - \psi$.

For example, if P, Q, R are the circumcentres of the triangles, $\theta = 2\alpha$, etc., and PQR has angles α, β, γ; while the triangle of orthocentres has angles $\pi - 2\alpha$, etc., and is therefore similar to the pedal triangle of any of the given similar triangles.

If P, Q, R are the minimum distance points of the triangles, so that $\theta = \phi = \psi = \frac{2}{3}\pi$, the triangle of the isogonal conjugates is

equilateral. If P, Q, R are the incentres of the triangles, $\theta = \frac{1}{2}(\pi + a)$, etc., and PQR has angles $\frac{1}{2}(\pi - a)$, etc.

The Brocard points of the triangles give triangles which are similar, but not directly similar, to ABC.

It is to be noted that the angles of the triangle PQR are independent of those of ABC, and hence applying the result to the triangle $A_2B_1C_1$, and so on, it will be found that, say, $P_2Q_1P_1R_1$ is a parallelogram, and that points P_n, Q_n, R_n similarly placed in the triangles $A_nB_nC_n$ form a nest of triangles, as in the case of equilateral triangles, but having angles $\theta - a$, $\phi - \beta$, $\psi - \gamma$, instead of being equilateral; while the triangles P_n', Q_n', R_n' have angles $\pi - \theta$, $\pi - \phi$, $\pi - \psi$, being independent of both A, B, C and a, β, γ.

C. Isosceles triangles having the same base angle.

9. If BCA_1, CAB_1, ABC_1 are isosceles triangles with base angles θ described outwards on the sides of any triangle ABC, and if X_1, Y_1, Z_1 are their orthocentres, we may prove that AA_1, BB_1, CC_1 meet in a point (P_θ) and that AX_1, BY_1, CZ_1 meet in a point ($P_{\frac{1}{2}\pi-\theta}$): also that Y_1Z_1 is perpendicular to AA_1, etc., and B_1C_1 to AX_1, etc., so that the angles of the triangle $A_1B_1C_1$ are equal to those between AX_1, BY_1, CZ_1; and the angles of the triangle $X_1Y_1Z_1$ are equal to those between AA_1, BB_1, CC_1. Further, we may prove that $Y_1Z_1 = AA_1 \cot\theta$, etc., and that $B_1C_1 = AX_1 \tan\theta$, etc. We may also prove that all the triangles $A_rB_rC_r$, $X_rY_rZ_r$ have the same centroid as ABC, and that the middle points of the sides are associated as in § 5.

It will be found that CB, Y_1Z_1, B_1C_1 meet in a point, the orthocentre of the triangle $A'A_1'X_1'$, and similarly for the other sets of corresponding sides. Hence ABC, $A_1B_1C_1$, $X_1Y_1Z_1$ have a common axis of perspective, so that the centres of perspective taken in pairs are collinear, namely, P_θ, $P_{\frac{1}{2}\pi-\theta}$, and O, the circumcentre of ABC. Also $OP_\theta : OP_{\frac{1}{2}\pi-\theta} = \Delta_1 : \delta$, where δ is the area of the triangle whose sides are AA_1, BB_1, CC_1.

The triangle $X_2Y_1Z_1$ will be found to be an isosceles triangle of base angle θ, so that the successive triangles of orthocentres are formed by the same rule as those of the vertices.

Using a figure similar to Fig. 2, we may prove that

$$\Delta_1 = \delta - \Delta \cot 2\theta \tan\theta.$$

The consideration of the area of $A_nB_nC_n$ leads to a difference equation, the solution of which is

$$\Delta_n = A\{\tfrac{1}{2}(t\sqrt{3}+1)\}^{2n} + B\{\tfrac{1}{2}(t\sqrt{3}-1)\}^{2n};$$

where $t = \tan\theta$, and A and B depend on Δ and δ.

10. The locus of P_θ is a conic passing through A, B, C and $P_{\frac{1}{2}\pi}$, the orthocentre of ABC, and is therefore a rectangular hyperbola. It also passes through P_0, the centroid; $P_{\frac{1}{3}\pi}$, $P_{\frac{1}{6}\pi}$, $P_{\frac{1}{4}\pi}$, three important special cases, and through $P_{-\frac{1}{3}\pi}$, $P_{-\frac{1}{6}\pi}$, $P_{-\frac{1}{4}\pi}$, where the minus sign refers to describing the triangles inwards. This makes 11 important points. The rectangular hyperbola passes also through the orthocentres formed by any three of these points, *i.e.* through about 160

additional points, since ${}^{11}C_3 = 165$. Again, the nine-points centre of any one of these triangles passes through the centre of the hyperbola, so that again some 160 circles all pass through this point. Further, the nine-point centre, the circumcentre and the symmedian point form a self-conjugate triangle with respect to the hyperbola.

It is further true that the locus of the isogonal conjugate of P_θ with respect to ABC is OK, where O is the circumcentre and K the symmedian point; and that the envelope of the common axis of perspective of ABC, $A_1B_1C_1$ and $X_1Y_1Z_1$ is a parabola inscribed in ABC. These properties afford a number of very simple exercises in trilinear coordinates.

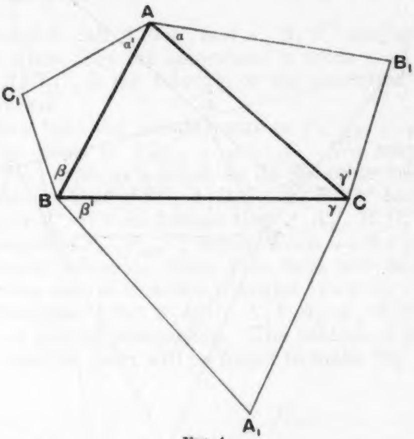

FIG. 4.

D. *The general case.*

11. Taking any triangles whatever described outwards on the sides of a given triangle, Ceva's theorem gives the following condition that AA_1, BB_1, CC_1 should be concurrent:

$$(\cot A + \cot \alpha)(\cot B + \cot \beta)(\cot C + \cot \gamma)$$
$$= (\cot A + \cot \alpha')(\cot B + \cot \beta')(\cot C + \cot \gamma');$$

and the set of necessary and sufficient conditions that this may be true for all values of A, B, C, subject to $A + B + C = \pi$, is $\alpha' = \alpha$, $\beta' = \beta$, $\gamma' = \gamma$. When this is so, call the point of concurrence $P_{\alpha\beta\gamma}$. The centroid of ABC is obviously P_{000} and the orthocentre P_{ABC}. The incentre is $P_{\frac{1}{2}(\pi - A),\, \frac{1}{2}(\pi - B),\, \frac{1}{2}(\pi - C)}$. It is easy to prove that the Brocard points are P_{CAB} and P_{BCA} respectively.

12. We may find the sides and angles of $A_1B_1C_1$ in terms of the elements of ABC and α, β, γ; in particular, we may find in what cases $A_1B_1C_1$ is equilateral. The relevant formulae are

$$\frac{a_1^2 \sin(\gamma + \alpha)\sin(\alpha + \beta)}{2\Delta \sin(\beta + \gamma)} - \frac{\Delta_1}{\Delta} = \frac{\lambda \sin(A + 2\alpha)}{\sin A}, \text{ etc.,}$$

where

$\lambda = -2 \sin \alpha \sin \beta \sin \gamma \cos (\alpha+\beta+\gamma)/\sin (\beta+\gamma) \sin (\gamma+\alpha) \sin (\alpha+\beta)$.

When $a_1 = b_1 = c_1 = x$, $\Delta_1 = x^2\sqrt{3}/4$, and we obtain

$$\left\{\frac{\sin (\gamma+\alpha) \sin (\alpha+\beta)}{\sin (\beta+\gamma)} - \frac{\sqrt{3}}{2}\right\} \frac{\sin A}{\sin (A+2\alpha)} = \text{two similar expressions.}$$

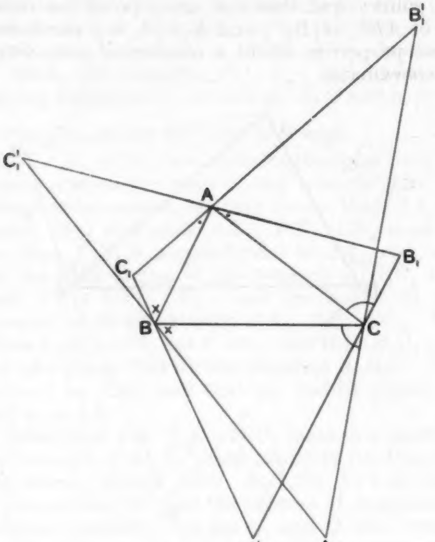

FIG. 5.

When $\alpha+\beta+\gamma = 2\pi/3$, these equations become

$\sin A/ \sin(\frac{1}{3}\pi - \alpha) \sin(\frac{1}{3}\pi + \alpha) \sin(A+2\alpha) =$ two similar expressions, two sets of solutions of which are $\alpha = \frac{1}{3}(\pi - A)$, etc., and $\alpha = -\frac{1}{3}A$, etc., the two cases of Morley's theorem concerning the trisectors of the external and internal angles of a triangle.

The calculation of the area of the nth triangle leads to a third-order difference equation, the solution of which is

$$\Delta_n = A_1 x_1^n + A_2 x_2^n + A_3 x_3^n ;$$

where A_1, A_2, A_3 depend on the original triangle and the first two to be constructed; and x_1, x_2, x_3 are the roots of the equation

$$(x+\lambda)\{(x-\lambda)^2 - x\} = 0,$$

where λ is the same as before. When $\alpha+\beta+\gamma = \pi$, $\lambda = 2$; and the equation becomes $(x+2)(x-1)(x-4) = 0$.

When $\alpha+\beta+\gamma=\frac{1}{2}\pi$, $\lambda=0$, and all the triangles are equal in area to the first one constructed. In this case A_1, B_1, C_1 are the incentres of directly similar triangles of angles 2α, 2β, 2γ; so that the angles of $A_1B_1C_1$ are $\frac{1}{2}\pi-\alpha$, $\frac{1}{2}\pi-\beta$, $\frac{1}{2}\pi-\gamma$, by § 8.

It then follows by simple geometry that $A_2B_2C_2$ is congruent to $A_1B_1C_1$, and that the succeeding triangles coincide in turn with $A_1B_1C_1$ and $A_2B_2C_2$.

13. Let BC_1, B_1C meet in A_1'; CA_1, C_1A in B_1'; AB_1, A_1B in C_1'. Then $B_1'AC=BAC_1'=\pi-A-\alpha$, etc., so that AA_1', BB_1', CC_1' meet in a point $P_{\alpha\beta\gamma}'$. Now, by construction, A_1, A_1'; B_1, B_1'; C_1, C_1' are isogonal conjugates of ABC. Hence so also are $P_{\alpha\beta\gamma}$ and $P_{\alpha\beta\gamma}'$.

It is convenient to call $A_1B_1C_1$ and $A_1'B_1'C_1'$ conjugate triangles. The only case when they are coincident is when $\alpha=\frac{1}{2}(\pi-A)$, etc., so that then $A_1B_1C_1$ is the triangle of the excentres of ABC and $P_{\alpha\beta\gamma}$ is the incentre.

We now have that the circumcentre is $P_{\pi-2A,\ \pi-2B,\ \pi-2C}$, and the symmedian point is $P_{\pi-A,\ \pi-B,\ \pi-C}$. We may prove that CB, B_1C_1, $B_1'C_1'$ meet in a point, as do the other two sets of corresponding sides, so that ABC, $A_1B_1C_1$, $A_1'B_1'C_1'$ have a common axis of perspective. It then follows that A_1A_1', B_1B_1', C_1C_1' meet in a point t (say) on $P_{\alpha\beta\gamma}P_{\alpha\beta\gamma}'$; while, when $\alpha+\beta+\gamma=\pi$, the case of directly similar triangles, these four lines will be found to be parallel. For the case of isosceles triangles $(\alpha=\beta=\gamma=\theta)$, it follows that the five triangles ABC, $A_1B_1C_1$, $X_1Y_1Z_1$, $A_1'B_1'C_1'$, $X_1'Y_1'Z_1'$ have a common axis of perspective. The centres of perspective of the triangles taken in pairs will be found to make Fig. 6.

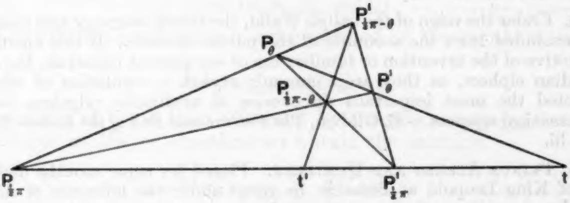

Fig. 6.

When $\theta=\frac{1}{3}\pi$, $t\to\infty$, and t' is the mid-point N of OH. Hence $P_{\frac{1}{3}\pi}P_{\frac{1}{3}\pi}'$ is parallel to $P_{\frac{1}{2}\pi}P_{\frac{1}{2}\pi}'$, and $P_{\frac{1}{6}\pi}P_{\frac{1}{6}\pi}'$ passes through the nine-points centre of ABC. When $\theta=\frac{1}{4}\pi$, the theorem asserts that $P_{\frac{1}{4}\pi}$, $P_{\frac{1}{4}\pi}'$, $P_{\frac{1}{2}\pi}$ are collinear.

There is a specially interesting connection between the case in which the triangles are described inwards for isosceles triangles and those described outwards, together with their conjugate triangles, but it is of extreme complexity. Further, it might be interesting to consider what happens when all the foregoing theorems are projected. *Ars longa, vita brevis.*

The President: I rather regret that Mr. Gibbins excluded algebraical methods from his very interesting paper until the end, when he gave us a set of difference equations. It seems to me to be worth mentioning that some of the results in the early part of the paper may be proved in an elegant manner by using the elementary algebra of vector-analysis.

For instance, if α, β, γ are the vectors from any origin to the vertices of the triangle ABC, then the vectors from this origin to the points X_1, Y_1, Z_1, which are the centres of the equilateral triangles described outwards on the sides of this triangle, are

$$\{\beta \exp(\tfrac{1}{6}\pi i) + \gamma \exp(-\tfrac{1}{6}\pi i)\}/\sqrt{3}$$

and two similar expressions; and the vector Y_1Z_1 is represented by $i\{\alpha + \beta \exp(-\tfrac{2}{3}\pi i) + \gamma \exp(\tfrac{2}{3}\pi i)\}$, which shows immediately that the triangle $X_1Y_1Z_1$ is equilateral. The theorems about three similar triangles described on the sides of a given triangle may be treated by the same method.

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963. In contemplating the series the first thing that strikes us is that the number of exogamous classes in a normal Australian tribe is always either two or a multiple of two; it is never an odd number. This raises a presumption that the organisation throughout is artificial and has been produced by successive and deliberate dichotomies of a previously undivided community, which was first divided into two, then in some cases by a second dichotomy into four, and lastly in other cases by a third dichotomy into eight. For had the origin of these exogamous divisions within a tribe been accidental, it is very unlikely that their number in all normal tribes should be either two or a multiple of two, never an odd number nor an even number indivisible by two.—J. G. Frazer, *Totemism and Exogamy* (Macmillan, 1910), vol. i. p. 273. [Per Prof. J. R. Wilton.]

964. Under the reign of the caliph Walid, the Greek language and characters were excluded from the accounts of the public revenue. If this change was productive of the invention or familiar use of our present numerals, the Arabic or Indian ciphers, as they are commonly styled, a regulation of office has promoted the most important discoveries of arithmetic, algebra, and the mathematical sciences.—E. Gibbon, *The Decline and Fall of the Roman Empire*, chap. lii.

965. Prince Albert and Quetelet. Placed for some months under the care of King Leopold at Brussels, he came under the influence of Adolphe Quetelet, a mathematical professor, who was particularly interested in the application of the laws of probability to political and moral phenomena; this line of inquiry attracted the Prince, and the friendship thus begun continued till the end of his life.—Lytton Strachey, *Queen Victoria*, chap. iv. § 1.

966. I am very sorry that I did not write to you yesterday, but my excuse must be that I forgot, as I was bothering myself, in company with some other members of my species, with a variety of abstruse and uninteresting mathematical problems, which notwithstanding the great good it is supposed I shall derive from them hereafter, are at the time, to say the least of it, dry. . . .—H. H. Asquith in a letter to his sister Eva, quoted in Spender and Asquith, *Life of Lord Oxford and Asquith*, vol. i. (1932), p. 28. The authors of the book add this heading to the letter: " Dated 20 May, 1861—Ætat 9 (but the real date is possibly 1865—Ætat 13) ".

MAYER'S METHOD OF SOLVING THE EQUATION $dz = P\,dx + Q\,dy$.

By M. L. Cartwright.

It has been pointed out * that Mayer's method of solving the differential equation

$$dz = P(x, y, z)dx + Q(x, y, z)dy \qquad \text{.....(1)}$$

is quite general and only requires one integration, whereas the other general methods require two, or even three, integrations. The reason for this is rather obscure in most text-books ; and the proof of the validity of the method rests on an existence theorem which leads to a solution of an apparently different form. I shall prove here that the two forms are identical.

We first observe that Mayer's method is a particular case of the more usual one of replacing one variable by a constant. For suppose that $P(x, y, z)$, $Q(x, y, z)$ and their first derivatives are continuous in the neighbourhood of the point (x_0, y_0, z_0), and also that

$$P\frac{\partial Q}{\partial z} - Q\frac{\partial P}{\partial z} + \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} = 0 \qquad \text{.....(2)}$$

in that neighbourhood. Putting $y - y_0 = m(x - x_0)$, we have

$$dz = \{P(x, y_0 + m(x - x_0), z) + mQ(x, y_0 + m(x - x_0), z)\}dx$$
$$+ (x - x_0)Q(x, y_0 + m(x - x_0), z)dm. \qquad \text{.....(3)}$$

Mayer's method † amounts to putting $dm = 0$; the equation (3) then has a unique solution

$$z = \phi(x, x_0, y_0, m, z_0) \qquad \text{.....(4)}$$

which is continuous in some neighbourhood of the point (x_0, y_0, z_0), and

$$\phi(x_0, x_0, y_0, m, z_0) = z_0.$$

The solution of (1) is then obtained by putting $m = (y - y_0)/(x - x_0)$ in (4). Hence by Mayer's method we obtain the solution

$$z = \phi\left(x, x_0, y_0, \frac{y - y_0}{x - x_0}, z_0\right) \qquad \text{.....(5)}$$

which passes through the point (x_0, y_0, z_0).

The more usual method ‡ applied to (3) requires the solution of

$$\frac{dz}{dm} = (x - x_0)Q(x, y_0 + m(x - x_0), z). \qquad \text{.....(6)}$$

Since $Q(x, y_0 + m(x - x_0), z)$ and its derivatives with respect to x, m,

* See F. Underwood, *Math. Gazette*, XVII (1933), 105-111.

† See L. Bieberbach, *Differentialgleichungen*, 3rd edition, Berlin (1930), 278-9 ; de la Vallée Poussin, *Cours d'Analyse Infinitésimale*, II, 6th edition, Paris (1928), 299.

‡ See L. Bieberbach, *loc. cit.*, 276-278.

and z are continuous, a solution exists,

$$z=\psi(x, m, x_0, y_0, m_0, z_0), \qquad (7)$$

passing through and valid near the point (x_0, m_0, z_0). We put

$$z=\phi(x, m, x_0, y_0, \psi(x_0, m, x_0, y_0, m_0, z_0))$$
$$=F(x, m, x_0, y_0, m_0, z_0), \qquad (8)$$

and we choose the form of (4) valid near (x_0, m_0, z_1), where z_1 is such that

$$F(x_0, m, x_0, y_0, m_0, z_0)=\psi(x_0, m, x_0, y_0, m_0, z_0).$$

Then it has been shown that (8) is the unique solution of (3) which passes through (x_0, m_0, z_0) and is valid in some neighbourhood of that point. Putting $m=(y-y_0)/(x-x_0)$, we have a solution of (1) valid in the corresponding region of values of x and y.

We reconcile (5) and (8) by means of the following theorem.

Theorem. *If $P(x, y, z)$ and $Q(x, y, z)$ and their first derivatives are continuous in the region $|x-x_0|\leqslant a$, $|y-y_0|\leqslant b$, $|z-z_0|\leqslant c$, and if (2) is satisfied in this region, then $\psi(x_0, m, x_0, y_0, m_0, z_0)$ is independent of m; and so*

$$\psi(x_0, m, x_0, y_0, m_0, z_0)\equiv\psi(x_0, m_0, x_0, y_0, m_0, z_0)=z_0$$

and $\quad z=F(x, m, x_0, y_0, m_0, z_0)\equiv\phi(x, m, x_0, y_0, z_0)$

is the solution of (3).

This follows easily from the existence theorem for (6); for

$$\partial\psi/\partial m=(x-x_0)Q(x, y_0+m(x-x_0), z)$$

is satisfied identically in virtue of (7). Since $Q(x, y, z)$ is a continuous function of all its arguments, we have

$$\left(\frac{\partial\psi}{\partial m}\right)_{x=x_0}=0$$

for all values of m for which (7) holds, and the theorem follows immediately from this.

It follows from the theorem that (5) is the solution of (1), and we have a new proof of Mayer's method. The usual proof assumes the existence of a solution in the form of a surface

$$z=\Phi(x, y)$$

passing through the point (x_0, y_0, z_0); and then, by considering the intersection of the surface with the plane $y-y_0=m(x-x_0)$, we reduce (1) to an ordinary differential equation in two variables. By solving this, we obtain a family of curves depending on a parameter m, and, putting $m=(y-y_0)/(x-x_0)$, we get the equation of the surface formed by them.

It should be observed that Mayer's method is not valid if P and Q are discontinuous at (x_0, y_0, z_0). For suppose

$$dz=\frac{dx}{x}+\frac{dy}{y}; \qquad (9)$$

then $P=x^{-1}$, $Q=y^{-1}$ are discontinuous at $(0, 0, z_0)$, and, putting $y=mx$, we have

$$dz=\frac{2dx}{x}+\frac{dm}{m}.$$

The solution of (9) given by Mayer's method is

$$z=2\log x+\text{constant},$$

while the correct solution is

$$z=\log x+\log y+\text{constant}.$$

But neither solution is continuous at the origin, and so it is not to be expected that the use of existence theorems, valid near the origin, will lead to a correct result.

Some text-books * present the standard process in rather a different form, and it is often difficult to see whether the conditions for the existence theorems which are used are really satisfied near the point considered. For this reason I have been unable to establish the connection with Mayer's method in a satisfactory manner except in the form just given. M. L. C.

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967. Without losing myself in calculations, I believe I am safe in voicing the opinion that our efforts in Chess attain only a hundredth of one per cent. of their rightful result. Our education, in all domains of endeavour, is frightfully wasteful of time and values. In Mathematics and in Physics the results arrived at are still worse than in Chess. Is there a tendency to keep the bulk of the people stupid ? For governments of an autocratic type the foolishness of the multitude has always been an asset. Possibly, also the mediocre who happen to be in authority follow the same policy.—E. Lasker, *Chess Manual*, p. 337. [Per Mr. A. F. Mackenzie.]

968. The mathematics of Chess does not, it is true, solve the problem of comprehending the contents of Life, but it sets that problem in precise terms and points to a solution. There the leverage will be supported whence investigators will set scientific research into motion. The first step is always the essential one. With the law of the lever by Archimedes came statics, with the law of the falling stone by Galileo arose dynamics, and their course, though their entry into the world was so modest, led them to revolutionise all science and all modes of living irrespective of the obstacles that hatred and stupidity heaped into their path.—E. Lasker, *Chess Manual*, p. 340. [Per Mr. A. F. Mackenzie.]

969. That Mental power wins in Chess would be the firmest of his (a budding Chessmaster's) convictions. But we reply that we speak of the force of the pieces, the force of their cooperation. He would probably be somewhat stunned by this suggestion and keep his peace. But he would be right in asking, whether such a thing as that really and truly exists. In mathematical books one is likely to find two vectors (a, b) called forces with a demonstration of Newton's parallelogram of forces attached thereto. Mathematically the assertion will be true, though a couple of vectors is a logical thing and not a reality like muscular power.—E. Lasker, *Chess Manual*, p. 255. [Per Mr. A. F. Mackenzie.]

* See E. L. Ince, *Ordinary Differential Equations* (1927), 52-56 ; H. T. H. Piaggio, *An Elementary Treatise on Differential Equations and their Applications* (1931), 140-142 ; E. Goursat, *Cours d'Analyse mathématique*, Paris (1918), Vol. II, 572-574.

THE SLIDE RULE IN THE TEACHING OF LOGARITHMS AND INDICES.

By E. J. Atkinson.

A logarithm as generally understood and used by the ordinary student is the expression of a number as a power of 10. In order to multiply two numbers together he adds the logarithms of these numbers and takes as the result the anti-logarithm of this sum. Similarly for division he takes the anti-logarithm of the difference of the logarithms of the two numbers.

As a means of introduction of the subject to a class it is common to set each boy to plot the graph $y=10^x$, between the values of $x=0$ and $x=1$. For this purpose the intermediate values selected are those most easily obtained by the direct arithmetical means of square root, namely $x=\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$. The product of the values obtained for $10^{\frac{1}{2}}$ and $10^{\frac{1}{4}}$ gives the value of $10^{\frac{3}{4}}$, and similarly by appropriate products the values of y for $x=\frac{3}{8}$, $\frac{5}{8}$ and $\frac{7}{8}$ are derived, giving the following table :

x	0	0·125	0·25	0·375	0·5	0·625	0·75	0·875	1·0
10^x or y	1	1·334	1·778	2·371	3·161	4·217	5·623	7·499	10

From the graph thus obtained the boy reads that $2=10^{0.3010}$ and that $3=10^{0.4771}$. He follows this with $2\times 3=10^{0.3010+0.4771}=10^{0.7781}$, and returning to his graph sees that when $x=0{\cdot}7781$, y is 6, showing that $3\times 2=6$. Also $\dfrac{6}{2}=\dfrac{10^{0.7781}}{10^{0.3010}}=10^{0.7781-0.3010}=10^{0.4771}=3.$

Henceforward the general advance is upon the line of logarithmic computation, and very many students who are extraordinarily familiar with logarithmic computation are quite unable to use the slide rule even for the simplest operations of multiplication and division.

The slide rule is a mechanical device by which these operations of multiplication (or division) are performed by adding (or subtracting) indices. The indices are represented by lengths, and thus the process becomes one of adding (or subtracting) lengths.

With the graph $y=10^x$ already plotted, and for convenience the scale for x being 10″ to represent unity, the making of a slide rule is a comparatively simple exercise.

When $y=1$, $x=0$, thus we have our starting point or zero. The transference of the graph values for corresponding values of x and y becomes a matter of projection on to a narrow strip of paper, and at once we have a scale with 1 at the beginning at zero distance and the successive numerals at distances from the zero point proportional to their logarithm to the base 10. If a second strip similarly marked be placed against the first, the operation 2×3 can be performed practically and mechanically, for log 2 represented on Scale I by 3·01″ is added to log 3 represented on Scale II by

4·771″, and the result actually at 7·781″ on Scale I shows the answer 6. Similarly for division, from log 6 represented on Scale I by 7·781″ take log 3 represented by 4·771″ on Scale II, and the result shows 2 represented on Scale I by 3·01″. The interest having been aroused, the class will wish to try other numbers, so points on the scales for 1·1, 1·2 and other intermediate points will become necessary.

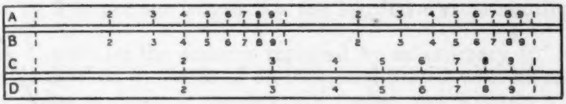

FIG. 1.

In practice it is found to be tedious and often confusing, especially when taking the decimal values, to project upon the x-axis for all the values of y which it is desired to mark. To obviate this and at the same time to gain greater accuracy the strip of paper upon which the scale is to be marked is moved, keeping its length parallel to the x-axis and the zero line on the y-axis. By this means it is possible to mark the strip at the point at which the graph meets it to record the corresponding value of y. Two strips thus similarly marked from 1 to 10 on this 10″ scale provide the C and D scales for our Slide Rule.

The accompanying graph (p. 110) shows the strip in the position to mark 5·6.

Log $10^2 = 2$ log 10. If we plot now the graph $y = 10^x$ on a scale one half of that of the first graph, namely 5″ to represent 1 unit for x, we shall be able to obtain scales which length for length will show numbers which are the squares of the numbers on Scales C and D. To make up the 10″ on this new scale the 5″ scale is marked twice, the second set of marks showing numbers which are 10 times those of the first set.

Two strips are marked with this double 5″ scale and are known as the A and B scales. It now remains to place these four scales in position to have the common or Mannheim form of Slide Rule. Scales A and B and Scales C and D move against each other to add or subtract lengths representative of the indices or logarithms. The Scales B and C are marked upon a central strip which may slide between the Scales A and D marked upon the stock or fixed portion.

Boys are keen to make things, and while some will make instruments in wood others will make good models in cardboard or of stiff paper. The wood or cardboard is cut 11″ long to accommodate the 10″ scale. In the case of the paper the sliding portion should be made more than 11″ long to enable the ends to pass through slits in the stock portion that they may be held in position when in use. The boys should be encouraged to make a cursor of thread or scratch on mica to assist reading and alignment.

The scales are so fixed that when in the zero positions the ends are in line. It is then seen that the numbers in A and B are the squares of the numbers underneath on C and D, thus squares and square roots can be read directly.

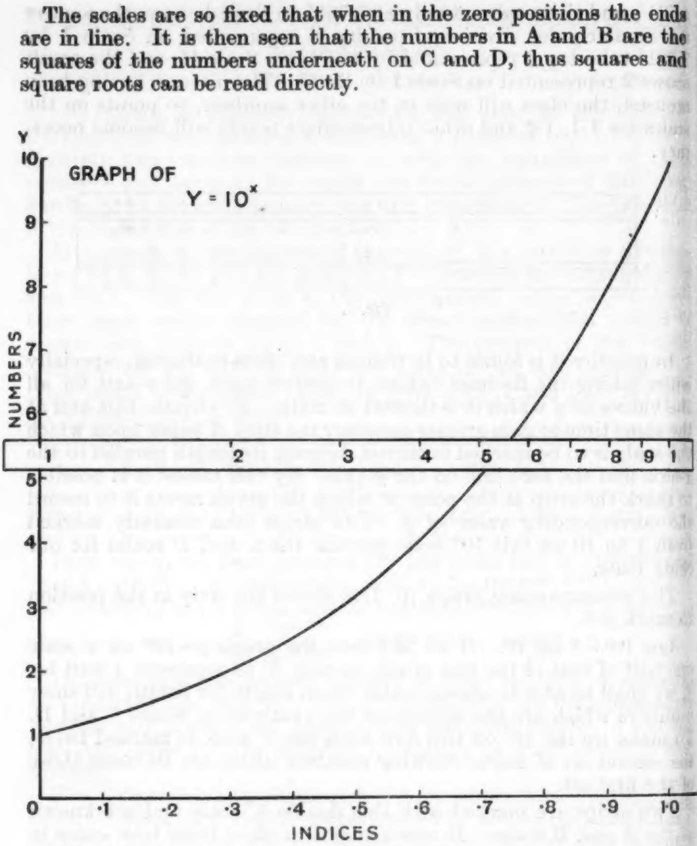

FIG. 2.

Log $x^{\frac{1}{3}} = \frac{1}{3} \log x$; so on our slide rule to find the cube root of a number the length representing its logarithm must be divided into three parts. This is done by adjusting the slider so that the number on Scale D against 1 on C is the same as the number on B against the number on A of which the cube root is required. If, for instance, the cube root of 64 is sought, the scales when adjusted will show 4 on D against 1 on C and 4 on B against 64 on A. Incidentally it will be seen that the 1 on B is against 16 on A. Closer examination will reveal that 4 on A is one-third of the distance of 64 from 1, and 16, which is 4^2, is two-thirds that distance, or that the distance 1 to 64, namely log 64, is divided by 3 to find the cube root.

A more simple but perhaps not so convincing method of finding the cube root is to take out the sliding portion and reverse it so that Scale C is against Scale A. With the scale readings now in reverse order the 1 on Scale C is placed against the number on A of which the cube root is required and it will be seen that at one point the number on Scale C is the same as that on Scale A. This number is the cube root which is being sought. Again, it will be observed that the length representing the number (*i.e.* its logarithm) has been divided by 3 to get the length (*i.e.* the logarithm) representing the cube root.

If the C scale on the slider is replaced by an ordinary 10″ ruler, preferably marked in tenths of an inch, and placed with its zero at the beginning of Scale D the boy will realise, not without a little surprise at first, that the reading on the ruler gives the logarithm of the number recorded on Scale D. Further, if Scale B is replaced by a scale 10″ long marked in half-inches, subdivided into 10 equal parts (*i.e.* showing 20″ but on half-scale), it will be possible to read on this scale not only the logs. of the numbers on Scale A, but also the characteristic for the numbers 10 to 100. These afford further incidents to bring home to the boy or student that the slide rule scales are but the representations of logarithm values by lengths.

Thus by a comparatively simple exercise we have the principles of indices and logarithms practically developed and applied. Further, a new interest is opened up in the matter of the Slide Rule, of which there are many variations. E. J. A.

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970. Selections from *Pepys' Diary*, 1661-2.

Jan. 10. To White Hall and there spoke with Sir Paul Neale about a mathematical request of my Lord's to him, which I did deliver to him, and he promised to employ somebody to answer it—something about observation of the moone and stars, but what I did not mind.

July 4. Comes Mr. Cooper, mate of the *Royal Charles*, of whom I intend to learn mathematiques, and do begin with him today, he being a very able man, and no great matter, I suppose, will content him. After an houre's being with him at arithmetique my first attempt being to learn the multiplicacion table: then we parted till tomorrow.

July 7. Comes Mr. Cooper: so he and I to our mathematiques.

July 9. Up by four o'clock, and at my multiplicacion-table hard, which is all the trouble I meet with at all in my arithmetique.

July 11. Up by four o'clock, and hard at my multiplicacion-table which I am now almost master of.

July 12. At night with Cooper at arithmetique.

July 18. Comes Cooper for my mathematiques, but, in good earnest, my head is so full of business that I cannot understand it as otherwise I should do. [Per Mr. B. E. Lawrence.]

971. Prince Nikolay Andreivitch Bolkonsky to his daughter Princess Marya: "Mathematics is a grand subject, madam. And to have you like the common run of our silly misses is what I don't want at all. Patience, and you'll get to like it." He patted her on the cheek. "It will drive all the nonsense out of your head."—Tolstoy, *War and Peace*, p. 105 (Heinemann, 1925). [Per Mr. F. Bowman.]

THE GEOMETRICAL NOTIONS OF YOUNG CHILDREN.*

By Mrs. E. M. Williams.

Perhaps my title will be thought to savour too much of the nursery, but I should like to reassure you. I have no intention of boring you with stories of my own children, although they have been responsible for directing my thoughts from my former pupils in Secondary Schools to the untaught little people whose ideas are so instructive.

All teachers of Mathematics are familiar with the extraordinarily wide divergences in mathematical ability even among the most intelligent pupils, and are driven to wonder at times of what strands this ability is woven; for it is no longer possible to regard it as a mysterious magic gift, the one single quality which makes all branches of mathematics equally easy to the possessor. Poincaré, for example, in his brilliant descriptions of his own processes of discovery and their comparison with those of others has shown how diverse these processes can be. There is the familiar distinction which he makes between the analyst and the pure geometer; and the psychologists have verified in their experiments how independent of one another are abilities in calculation, in the symbolic operations of algebra, and in the solution of problems in geometry. Now arithmetic has quite a large literature of its own; McLellan and Dewey, Margaret Drummond and Thorndike, among others, have written on its psychology. Algebra, too, has had its investigators, but when we turn to geometry the roll grows less. I should like to examine very briefly what has already been done in the analysis of geometrical ability.

Perhaps the first place should be given to Ernst Mach, whose work on Mechanics as on Geometry has started so many on a new line of thought. As you will remember, in *Space and Geometry* (1906), he sets out the dependence of geometry on our physiological structure and our experience of actual space; but his work is based entirely on his own memories and introspection; it has a subjective basis and therefore fails to satisfy the demand of the modern psychologist for objective data, the results of experiments on or observations of individual reactions to spatial experience or geometric situations. For our present purpose the significance of his work lies in the fact that he traces the growth of the science of geometry from our earliest attempts at movement and at adjusting various muscles, such as those of the eye, to the changed positions of interesting objects. He suggests to us, therefore, that we should study this adaptation of the child to spatial conditions as one factor at least in geometrical ability.

Writing three years later William Brown stated: " Each school subject has potentially a psychology as well as a logic, but introspection alone is insufficient to give us its data "; and he went on

* A paper to the London Branch, 11th November, 1933.

to investigate objectively mathematical "intelligence" as he called it. But his method will not serve our purpose with young children; it presupposes some training in school mathematics. What actually he did was to set questions of the normal school type: a proposition to write from memory, and a simple rider to solve, for example; but he marked for special traits or processes, such as memory of a general truth and the power to apply it in an individual case; the power to recognise general truths in a special case, etc. You will see at once how subjective this analysing into processes must be and how little it can help with children who have had no formal teaching in geometry. In passing it is perhaps worth mentioning that the faculty he found to be of greatest value was the one I first quoted.

A more ambitious attempt was made in 1918 by Miss Rogers of Columbia University; she used a large number of specially devised tests, some original, with the avowed object of finding as many as possible of the different factors which make up mathematical ability, whatever they might be, and so making a selection of tests which would diagnose that ability with fair accuracy. Besides arithmetical problems and algebraic equations her material included simple riders on given geometrical facts, tests of symmetry and superposition, exercises on cross-sections of solids and the definition of certain given figures. Her conclusions are, however, rather disappointing, for two reasons—her tests imply previous training in mathematics, which rules them out for pre-Secondary School testing as a rule; and also she gives no analysis in the end of the strands of which the general mathematical aptitude is composed. Instead she standardises her tests so that they can be used as a means of finding the capacity for mathematics of a pupil who is already being taught and examined in the school: useful sometimes when we have reason to suspect laziness or a lack of sympathy with the teacher. It is interesting that Miss Rogers also found that one of the tests which were most significant of real ability was the solution of problems using a selection of given facts.

There are two lines of enquiry that bear on the problem: the child's power of reasoning—since geometry is a science—and his power of dealing with the spatial properties of objects—since geometry is the science of space. As we had Professor Burt himself here last session to talk to us, I need only refer to the way in which he has traced the development in complexity and the growth in sustaining power of the child's capacity for reasoning and the excellence of his graded tests. Piaget of Geneva has taken a different line of investigation; in numerous interviews with children he has studied the way in which they actually reason about matters of their daily experience—such questions as that of relationship within the family. He makes two interesting points: first, that the child under seven is not aware of logical implications; and then that the child from seven to eleven is incapable of formal logic, that is, that he must first be convinced of the truth of his premisses before he will admit the validity of his conclusion. He cannot say, "*If*

this is true, *then* that must be true ". Teachers of geometry will at once recall the difficulties of the hypothetical construction. Piaget has put it in this way—the child has a logic of action but not yet a logic of thought. It is just this belief that makes the second line of enquiry so interesting—the investigation of the child's capacity for understanding the spatial properties of objects. We may expect to see him *act* logically even if he cannot yet *argue* logically. The method that has most often been employed has been that of form-boards, *i.e.* of boards with depressions of various geometric shapes into which insets can be made to fit. Unfortunately these have been devised all too frequently as a test for mental defectives and their spatial significance ignored ; so far as I know, no one has yet used them to trace the child's improvement in the grasp of spatial relations, or the widening of his geometrical ideas. And this, I think, is what geometry needs if the theory of the method of teaching it is to be put on the surest foundations ; that is, we need a *genetic* study of the growth of geometrical ability. We all know how much the teaching of reading and elementary arithmetic owes to our utilising the child's natural method of approach and his normal stages of development, and it is one of the aims of the researches into special aptitudes that are now being conducted at the Institute of Education (with the co-operation of the Education Department of King's College) that they may lay the foundation of a scientific as opposed to an empirical study of methods of teaching.

These researches have another use—the diagnosing of special abilities ; and here we are up against one of the difficulties of selecting for admission to Secondary Schools. How can we detect whether the child is going to show the capacities for mathematics and science and foreign languages that are required for the normal Secondary School course ? Those who have read Professor Valentine's book, *The Reliability of Examinations*, will remember that his enquiry shows how little the order of candidates in the entrance examination corresponds with their order in any of the examinations of the Secondary School after the first year. Is this because the present entrance tests do not select the children of greatest ability ? Or is it that they have no relation to the new subjects that the child must learn ? Several solutions of the problem could be suggested : intelligence tests seem to reinforce the reliability of the usual entrance tests in arithmetic and English (though one hears complaints that these are being crammed into some elementary school children, a very unsuitable diet) ; and, in addition, an attempt could be made to devise methods of testing which would be prognostic of later success in mathematics or languages. This can only be done by using the tests over a period of years and finding their actual correspondence with the achievements of the candidates in their school career. But to devise tests that will carry conviction and will select an untutored aptitude we need this analysis of the ability into its components, the order of their appearance and the rate of their growth in the normal as well as in the clever child.

As an example of what is being done I should like to tell you about a study of the child's notion of symmetry. One should start, of course, with the infant, and in addition not scorn any available facts about animals. The first movement of the baby which has a bearing upon it is the turning of its head in the first few weeks of life in response to a sound or a bright light on right or left. Then it begins to sit up and to have sensations due to its own axis of symmetry, sensations which are extended as he learns to walk. But how little can he distinguish between left and right! Baldwin tested his children at ten months with coloured balls and found that they reached with either hand quite indifferently; and this bears out our own experience that very young children use both hands without prejudice to either. We are familiar, too, with the child's confusion between 3 and Ɛ and the trouble some children have in first writing the numbers from 10 to 20, writing Γ1 for 17. In fact it is at age six that Binet puts the test to distinguish the right hand from the left. But the vertical posture of the human helps the child. He never, for instance, would make the amusing mistake of Köhler's apes; they wanted to reach some fruit placed just beyond their reach outside the bars of their cage. Now when the fruit had been placed on the bars overhead they had learned to get it down by piling up boxes and climbing on to them; so now they dragged the boxes to the edge of the cage to help them reach the fruit outside. Man, on the other hand, develops a very early knowledge of the significance of his vertical axis of symmetry.

These special tests on symmetry were planned for children from five to eight years of age, that being the period in which the child makes this stride of being able to tell left from right with certainty. There are two parts, closely related. First is the set of formboards, four of them. The first has three pairs of similar insets, each pair having sides in the ratio of 4 to 5, and the shapes being simple symmetrical ones, circle, square, triangle. The second set, triangles, rectangles and pentagons, have only one axis of symmetry, kept apparently vertical, the sides preserving the ratio of 4 to 5. In the third, the corresponding shapes have different orientation; the fourth is like the first, but with more difficult shapes, regular hexagons, pentagons and octagons. This is needed because the first test is not altogether reliable because of its strangeness to the child; that is why it was made so easy. The second part of the test was chosen as a check and an amplification of the first part. It consists of a circle of cardboard made to rotate against a board of contrasting colour; bright red and blue were chosen to attract the little people. From the circle were cut out shapes like those on the middle boards, so that the holes showed through bright red; the insets had to be replaced by the child in the seven successive positions of the wheel. An extra shape was put in—a square—to prevent there being too obvious a relationship between the different positions of the cut-outs on the circle.

The children were told that we were going to see who was the quickest, and were timed with a stop-watch for each formboard and

for each position of the wheel. Each child had to be tested separately, and was also given an intelligence test as a means of comparison of general ability. The most surprising thing was the enjoyment of the examinees. All the children in the school wanted to come, and very few indeed were the occasions when lack of interest appeared.

It is of importance to know *how* the child selected the right hole for the inset, so careful notes were made of his methods and of his mistakes and failures. These are more instructive than his actual score in time taken.

Some of the more interesting facts that have come out of it are these: the child matures very rapidly during the period at this kind of work, as is shown by the following scores, calculated as proportional to the reciprocals of the times taken:

Age	Mean Scores on boards	Mean Scores on wheel
5	38	20
6	41	25
7	51	31
8	58	38
Range of scores	27—75	7—64
Mean score	47	29

Intelligence also plays a considerable part, but its influence is not as great as that of age; the correlations work out as ·61 for score with age and ·46 for score with intelligence. The latter figure is rather surprisingly low, but this is due to the most interesting individuals in the tested group; they are children of rather low intelligence, varying from 71 to 86 out of 200, and of various ages, but all of whom showed some definite imaginative ability; either they were unusually good at drawing or they showed some special talent in the interpretation of pictures, which is one of the Binet tests, or they showed some special interest in shape and pattern; *e.g.* one of them, instead of comparing the weights of the boxes as she was asked, said, "I can draw round these, can't I?" Another could not do the writing he was set, but said hopefully, "But I can draw you a picture", and he produced a quite remarkable drawing of a frog drinking from a mug. These cases are likely to be the most instructive, so they are being enquired into more closely; they remind me of the children who do very well at the beginnings of geometry, especially the practical part, but fail to keep up when more logical work is begun; but so far that is mere speculation and remains to be proved.

Another result that may interest you concerns the procedure that the children adopted; there were three markedly different modes: the first was mere trial and error, that is, they took the inset and moved it about the board or above it until it fitted. This was the method most used by the five-year-olds and by those with poor scores; it was characteristic of these people to talk aloud about the work, sometimes rather amusingly; one little boy found the

pentagons hard; "You have made that corner too big", he said; and "You've done that wrong". They had *seen* the insets taken out of their holes, yet several of them said that the inset would not fit anywhere. "There isn't a hole for this one." They would try most painstakingly to fit a single side or an angle, but with no system and with no awareness of the symmetry of the shape. The second way of handling them showed a definite perception of symmetry, but only when the axis was apparently vertical. These children picked up the inset, at once turned it till its axis was upright; often they turned their heads so that the hole had the same relative look; some tried to turn the wheel to get the same effect. A few of the five-year-olds used this method and usually it went with a moderate score. The third method was the one of almost immediate insight; no five-year-old used this way completely, but some achieved it on occasion and with the simpler shapes, but the highest scorers and the most intelligent used it, with perhaps an occasional reversion to one of the other types. For example, a child who was very successful with the triangles and the rectangles would fail with the pentagon; the reactions would vary from just taking another look at the inset to the most unintelligent fumbling; in some cases it would mean a relapse to fitting each side or angle; except with the most intelligent a failure to carry in the mind the wholeness of the one shape until it had been matched with the corresponding one seemed to cause a kind of disintegration, a breaking down into smaller units for comparison or a return to the primitive method of trial and error.

This contrasts very remarkably with the performance of the boy who made the record score. He was a lad of 8½ with the excellent I.Q. of 141. His times were for the boards: 12, 11, 20, 14 seconds; for the wheel: 21, 17, 18, 14, 14, 14, 14 seconds. The third board with the difference of orientation gave him trouble; he had one relapse to fumbling; when he came to the wheel he was at first a little slower, improving to the record time of 14 secs., which he maintained to the end; he showed that he had grasped the whole scheme.

An analysis of these different methods used by the children indicates a definite change in their attitude to symmetry; five-year-olds are hardly aware of it; they go on to notice it as a kind of balance wrapped up with their own possession of a plane of symmetry; later it can be distinguished as a relationship between the different parts of a figure, regardless of its position relative to the child. There is also a change in their capacity for holding or retaining a relation that they have perceived. At first the inset has to be brought quite close to the hole to be compared; it was carried all over the board in the search for the shape that matched. Later on as the symmetry is noticed the inset is turned to make the relation easy to carry in the mind; repeated comparisons are often necessary at this stage; finally, one glance gives the relation which is carried until the same relation is recognised in the depression into which the inset is at once fitted.

Many other facts could be quoted as observed in these tests, but perhaps enough has been said to show the kind of way in which development in geometric ideas can be traced. We need to go on to extend our knowledge of this one idea of symmetry through the succeeding years of 8 to 11; further tests are being used of greater difficulty, and so we can hope to reach a norm which will determine with certainty the children who at 11 have definite ability in mathematical processes. Naturally we do not rely on testing just one conception such as symmetry; we need to have all the essential ideas studied in a similar way, and a selection of tests made which will on the one hand give the most accurate forecast and on the other hand will be easy to administer to large numbers. Tests by doing are necessary for *little* children, but written and drawing tests are possible at 11.

Schemes of this kind are being worked out, and the co-operation of teachers in the schools will be very warmly welcomed, not only in giving facilities for the administering of tests and the keeping of records later, but also in giving us the results of their experience in selecting entrants for Secondary Schools. E. M. W.

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972. It is a sobering reflection that Clerk Maxwell should have lived in order that the remoter ether might vibrate to the strains of negroid music.—C. E. M. Joad, *Under the Fifth Rib*, p. 168 (1932). [Per Professor E. H. Neville.]

973. In Wahrheit aber sind Raum und Zeit, wie diese ganze Betrachtung überzeugend lehrt, wesengleich, und damit ist der Dualismus zwischen ihnen zu einer höheren Einheit verschmolzen.

Erst ziemlich spät ist die denkende Wissenschaft in dieser Richtung zur Klarheit gelangt, und die jetzt so viel besprochene Relativitätstheorie erst hat jene höhere Einheit von Raum und Zeit bis zum Ende durchdacht. Dass aber inspirierte Geister schon vor Jahrhunderten den Grundgedanken erfasst hatten, dafür sei unter vielen nur *ein* Zeugnis beigebracht, der schöne Vers des deutschen Dichters Angelus Silesius:

Die Rose, welche hier
Dein äussres Auge sieht,
Die hat von Ewigkeit
In Gott also geblüht.

—F. Auerbach, *Lebendige Mathematik*, 1929, p. 113.

974. We came to the residence of a Mr. Giddy whose son I remember to have seen at Oxford, a gentleman commoner of Pembroke College. He was then generally pointed out as a ralph, as the elegant phrase is for anyone who is not attentive to their dress. But this very ralph may chance to have his name remembered when the fine gentlemen who were so proud in their own fancied superiority are no more heard of, as he is generally esteemed a very good scholar and turns his studies chiefly to mathematics and possesses the greatest collection of books in that science I ever saw in a private library; to be sure his dress and address are not very prepossessing, but I am certain in real civility he is not at all deficient.—*B.M. Add. MSS. 28793*, The Rev. J. Skinner's Journal, November 1797.

For Gilbert (formerly Giddy) Davies (1767-1839), M.A., D.C.L., M.P. (1804-1832), F.S.A., and President of the Royal Society (1827-1830), see D.N.B. [Per Mr. A. T. Wicks.]

THE PILLORY.

THE following question was No. 7 in the Applied Mathematics paper of the Intermediate examination in Engineering of the University of London in 1922:

"Where and how does the external tractive force on a locomotive act? The horse-power developed by a locomotive going at 25 M.P.H. is 20, the weight of the locomotive is 40 tons and the resistance 7 lb. per ton; if the acceleration be constant, find the tractive force, the time taken and the distance gone from rest. The maximum horse-power that can be developed being 25, find the greatest distance that can be gone in three hours from rest."

With the numbers given in this question the locomotive takes about 85 minutes to get up a speed of 25 M.P.H., and it has then travelled 18 miles! A student would immediately think that he was wrong if he obtained such an answer and thereby waste valuable time in the examination. Quite apart from these ridiculous numerical answers, I deprecate the use of the term "tractive force" for the frictional force of the rails, and I would like to see questions of this type in which there has to be a discontinuity in the train's acceleration when it gets to its maximum speed banished from examination papers. Moreover, it should be definitely stated in the question that in the first part of the motion the engine is working with a constant force, and not at a constant power.

In another question in the same examination paper a machine is working at constant power, but the student is not told this.

Woolwich Polytechnic. H. V. LOWRY.

Examination for Entrance Scholarships and Exhibitions. King's group, Cambridge. Friday, 11th December, 1931.

B. 10. "A particle of mass m resting on the highest point of a fixed sphere of radius a and coefficient of friction $\frac{1}{2}$ is slightly disturbed and slides down the sphere in a vertical plane. Prove that when the radius to the particle makes an angle θ with the vertical the angular velocity, $\dot{\theta}$, of the radius is given by the equation

$$\frac{d}{d\theta}(\dot{\theta}^2 e^{-\theta}) = \frac{g}{a}(2 \sin \theta - \cos \theta) e^{-\theta},$$

and show that the normal reaction on the sphere at this instant is

$$\tfrac{1}{2} mg\,[3(\cos\theta + \sin\theta) - e^{\theta}]."$$

The first part of the question is right, and the second part "comes out" if it is assumed that $\dot{\theta}$ vanishes with θ. But this assumption is incorrect since the particle will rest anywhere between the positions $\theta=0$ and $\theta=\text{arc}\tan\frac{1}{2}$, and does not begin to move under gravity until the position of limiting equilibrium is past, that is, until θ is greater than arc tan $\frac{1}{2}$.

N. M. GIBBINS.

MATHEMATICAL NOTES.

1100. *A bibliographical note.*

The *Mathematical Gazette* has recently given considerable attention to the proposition: If the internal bisectors of the two base angles of a triangle are equal, the triangle is isosceles.* The following bibliographical data may therefore be of interest to some of the readers of the *Gazette*.

The proposition is almost a century old. It was communicated by Lehmus to J. Steiner in 1840. One may thus say with the poet: "Es ist eine alte Geschichte, doch bleibt sie immer neu". J. S. Mackay devoted a very interesting sketch to this question.† Numerous other bibliographical references may be found in M. Simon's book, *Ueber die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert*, pp. 131-133 (Leipzig, Teubner, 1906).

Here are some further additions: (i) *Journal de mathématiques élémentaires et spéciales*, 1880, p. 538. (ii) *Bulletin des sciences mathématiques et physiques élémentaires*, vol. IX (1903-4), p. 195. (iii) *American Mathematical Monthly*, vol. II (1895), pp. 157 and 189-191, problem No. 42; vol. XXIV (1917), p. 344. (iv) *Mathesis*, 1896, p. 154 (Barbarin); 1923, p. 337; 1930, pp. 55 and 97; 1932, p. 162. (v) *School Science and Mathematics*, vol. XXXIII (1933), pp. 781-783, problem No. 1283. See also J. Neuberg's note in *Mathesis*, 1907, p. 184.

A triangle may have two equal *external* bisectors without being isosceles. The triangle is then called "pseudo-isosceles". A good deal has already been written about this pseudo-isosceles triangle. (i) *Intermédiaire des mathématiciens*, 1894, pp. 70 and 149; 1895, pp. 101 and 169. (ii) *Mathesis*, 1895, p. 261; 1900, pp. 129 ff. (A. Emmerich); 1901, p. 24, note 1; 1902, p. 43, note 5; 1902, pp. 112-114 (G. Delahaye); 1925, pp. 316 ff. (iii) *Bulletin des sciences mathématiques et physiques élémentaires*, vol. IX (1903-4), p. 146, and vol. XII (1907-8), p. 22 (G. Fontené). See also J. Neuberg, "Bibliographie des Triangles Spéciaux", pp. 9-11 (Brussels, Ramlot, 1924. Reprinted from *Mémoires de la Société Royale des Sciences*, Series 3, vol. XII).

N. Altshiller-Court.

1101. *The Teaching of Logarithms.*

I have been astonished of late to find a number of otherwise enlightened teachers of mathematics deprecating the fact that boys of thirteen and fourteen in the elementary schools are being taught logarithms. That logarithms should be taught in the elementary school seems to me a very hopeful sign, and I hope that the time is not far off when every boy and girl of fourteen will be able to

* See *Math. Gazette*, XVI, July, 1932, p. 200, Note 1031; XVII, May, 1933, p. 122, Note 1069, and October, 1933, pp. 243 ff.

† "History of a Theorem in Elementary Geometry", *Proceedings of the Edinburgh Math. Soc.*, vol. XX (1901-2), pp. 18-22.

manipulate the useful tool that a knowledge of logarithms provides. In the secondary school, the use of logarithms can be taught very usefully at the beginning of the year below School Certificate, and is one of the few things that can be taught to dull boys and girls with any degree of success.

The old method of teaching logarithms is, however, very difficult for the dull or even mediocre child ; for example :

let $$2364 \times 689 = x,$$
then $$\log 2364 + \log 689 = \log x.$$

Associations must be direct and simple for the dull pupil, and, put in this way, the connection between the characteristic and the number of digits is not obvious, so it is mostly chance whether or not the child gets the correct characteristic for the logarithm, or the correct position of the decimal point in the anti-logarithm.

The following method is probably adopted by a large number of teachers, but I see a great many who still teach the other way.

I should begin logarithms without any preliminary lesson on indices other than the mention of them in multiplication and division. Reminding the class that $x^3 \times x^2 = x^5$, we have, since x is any number, $10^3 \times 10^2 = 10^5$. This adding of the indices may be used to shorten different kinds of multiplication by putting all the numbers in powers of 10.

$$10^1 = 10,$$
$$10^2 = 100.$$

What power of 10 is 20 ? It must be 10^{1+}, and so on. It seems then that fractional indices are needed, so we must discover a meaning for $10^{\frac{1}{2}}$, $10^{\frac{1}{4}}$, and so on. Then for 10^0, and for 10^{-1}.

Now plot a curve of $10^x = y$ for the values of x between 0 and 2, and then a more accurate one for values of x between 0 and 1, using this for a few examples, first in reading off the powers of 10 for different numbers, and later from which to work simple multiplication. By using this graph before the tables, the fact that only the whole number need change for numbers in which the same digits are in the same order is easily grasped. Thus, from the table ·14 and ·27 can be put in powers of 10 and their product found. Now for the product of 1·4 and ·027, the children readily see that ·14 being obtained from the table, 1·4 immediately follows, and similarly for ·027.

At the expense of using more paper and taking a little longer time, I advocate, very strongly, because I have taught both methods, keeping in the powers of 10 for the first term at least—in any case long enough for a child to say to himself in considering the logarithm of, say, 1789 : " This is more than 1000 but less than 10,000 and so is 10^{3+} ". Finally the whole process becomes mechanical, but if the habit gets rusty the explanation can easily be called into action, and the haphazard method of writing down the characteristic of the logarithm is replaced by a sure method of easy reasoning.

The objection to this " power of ten " method is sometimes raised

that children so taught do not acquire the right conception of logarithms to bases other than 10, but surely logarithms to other bases can be taught in the School Certificate year. In the first year of the advanced course, logarithms to base e would of course be taught, and a good deal of interest is aroused by the computation of log 2 and log 3 to base e from first principles, and hence the calculation of log 2 and log 3 to base 10. Later on, if there is time, the history of the discovery of logarithms is of great interest.

However, the tool of logarithms to the base 10 is such an extremely useful one that it should be in the hands of the elementary school boy or girl before he or she leaves school. It is within their scope of manipulation and, I venture to say, within the scope of their understanding also.

H. W. OLDHAM.

1102. *Unit torque, unit angular momentum and the hypothesis of homogeneity.*

In dynamics it is usual to define dynamical force, linear momentum and linear kinetic energy by $M\ddot{X}$, $M\dot{X}$ and $\frac{1}{2}M\dot{X}^2$, where M is a constant of proportionality which is called mass. We may make corresponding definitions in terms of angle.

We may suppose that torque, C (dynamical moment), is proportional to angular acceleration, $\ddot{\Theta}$, so that

$$C = I\ddot{\Theta}, \quad \text{.................................(i)}$$

where I is a constant of proportionality as yet undefined. We may then define angular kinetic energy and angular momentum by

$$\text{angular kinetic energy} = \tfrac{1}{2}I\dot{\Theta}^2, \quad \text{.................................(ii)}$$

$$\text{angular momentum} \quad = I\dot{\Theta},^* \quad \text{.................................(iii)}$$

Then by the geometric equalities †

$$\tfrac{1}{2}\dot{\Theta}^2 = \ddot{\Theta}\Theta,$$

$$\dot{\Theta} = \ddot{\Theta}T,$$

we get

$$\text{angular kinetic energy} = \tfrac{1}{2}I\dot{\Theta}^2 = I\ddot{\Theta}\Theta = C\Theta, \quad \text{............(iv)}$$

$$\text{angular momentum} \quad = I\dot{\Theta} = I\ddot{\Theta}T = CT. \text{..................(v)}$$

We next need to give I such a meaning as will connect the above definitions with the corresponding definitions associated with the linear variable. We use Euclid's proportionality connecting arc and

* All the symbols denote magnitudes ; for example,

$$\dot{\Theta} = \frac{d\Theta}{dT} = \frac{d\,(\theta\kappa)}{d\,(t\,\text{sec.})} = \frac{d\theta}{dt}\left(\frac{\kappa}{\text{sec.}}\right) = \dot{\theta}\left(\frac{\kappa}{\text{sec.}}\right),$$

where κ is the radian.

† The analogues of $\frac{1}{2}V^2 = AS$, $V = AT$, the initial velocity being supposed zero.

angle, and the result of differentiation with respect to time, namely

$$S = R\frac{\Theta}{\kappa}, \quad \dot{S} = R\frac{\dot{\Theta}}{\kappa}, \quad \ddot{S} = R\frac{\ddot{\Theta}}{\kappa}.$$

For a point moving in a circle, radius R, we have

linear kinetic energy = force × distance ;

$$\tfrac{1}{2}M\left(R\frac{\dot{\Theta}}{\kappa}\right)^2 = M\left(R\frac{\ddot{\Theta}}{\kappa}\right)\left(R\frac{\Theta}{\kappa}\right)$$

$$= \left(\frac{MR^2\ddot{\Theta}}{\kappa^2}\right)\Theta,$$

and this agrees with (iv) if $I = MR^2/\kappa^2$. So for this value of I the angular kinetic energy and the linear kinetic energy are identical. With this meaning for I, we must interpret the definitions of torque and angular momentum.

We have

$$C = I\ddot{\Theta} \quad \text{...(vi)}$$

$$= \left(\frac{MR^2}{\kappa^2}\right)\ddot{\Theta}$$

$$= M\left(\frac{R\ddot{\Theta}}{\kappa}\right)\frac{R}{\kappa}$$

= (force × radius)/radian

= (statical moment)/radian,

and

unit dynamic moment = (lb. ft.^{2})/(sec.^{2} κ) or (lbwt. ft.)/κ.

Also angular momentum $= I\dot{\Theta}$

$$= \left(\frac{MR^2}{\kappa^2}\right)\dot{\Theta}$$

$$= M\left(\frac{R\dot{\Theta}}{\kappa}\right)\frac{R}{\kappa}$$

= (linear momentum × radius)/radian

or = (moment of linear momentum)/radian,

and

unit angular momentum = (lb. ft.^{2})/(sec. κ) or (lbwt. ft. sec.)/κ.

There is usually confusion between unit torque and unit work when both are denoted by ft. lbwt. It is customary to denote the difference between the two concepts by a change in the order of the symbols ; the objection to this procedure is that the factors in products of magnitudes are commutative. This confusion is avoided when (ft. lbwt.)/κ is taken as unit torque. In statics, of course, the moment of a force is defined by

moment = λ (force) (perpendicular distance),

and no difficulty arises when λ is identified with the abstract number 1 because angle variation is not involved. In dynamics, angle variation is involved and the matter is different. In like manner, unit angular momentum, (ft. lbwt. sec.)/κ, is distinguishable clearly from unit action, ft. lbwt. sec. Nothing but obscurity can result from the identification of all physical constants of proportionality with the abstract number 1.

Other advantages of definitions (i), (ii) and (iii) are that the torque-angle, force-distance analogy is complete and that all equalities become homogeneous in angle as well as in length, time and mass.

V. NAYLOR.

1103. *Proof of a theorem in permutations.*

Let m, n and h be positive integers, where $m \geqslant n-h$. To find the number of different routes, composed of positive integral steps parallel to axes Ox, Oy, in passing from the point (0, 0) to the point (m, n) without *crossing* the line $y=x+h$. Of course, without this restriction, the number of routes is ${}^{m+n}C_m$. Let the point (m, n) be called P and let P_1 be its reflection in the line $y=x+h+1$, called the line l for brevity. Then P_1 has coordinates $(n-h-1, m+h+1)$; we will suppose $n \geqslant h+1$. Let $R_{m,n}$ be the number of routes required. Then

$$R_{m,n} = {}^{m+n}C_m - F_{m,n}$$

where $F_{m,n}$ is the number of routes from O to P which *cross* the line $y=x+h$, or, alternatively, which *meet* the line l in one or more points. Suppose any such route meets l for the first time in a point Q, then it is obvious that the reflection of the portion of this path from Q to P in the line l will pass through P_1. Hence

$$F_{m,n} \leqslant {}^{m+n}C_{m+h+1},$$

the number of unrestricted routes from O to P_1. Now all of the ${}^{m+n}C_{m+h+1}$ routes from O to P_1 cross l at least once. Suppose any of these routes crosses for the first time at Q_1. Then again it is obvious that the reflection in l of the portion of the path from Q_1 to P_1 passes through P.

Hence $${}^{m+n}C_{m+h+1} \leqslant F_{m,n}.$$

Thus $$F_{m,n} = {}^{m+n}C_{m+h+1}$$

and $$R_{m,n} = {}^{m+n}C_m - {}^{m+n}C_{m+h+1}.$$

In this, of course, h may be zero. Again if $n < h+1$, then

$$R_{m,n} = {}^{m+n}C_m,$$

as is obvious.

The theorem is not new; but this proof seems simpler and briefer than the majority of those on record.

J. PEACOCK.

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975. *Clue:* Double the smallest Stock Exchange fraction. *Word meant:* Sixty-fourth.—Crossword in *Morning Post*, 6th July, 1933. [Per Mr. I. FitzRoy Jones.]

REVIEWS.

Reihenentwicklungen in der mathematischen Physik. By J. LENSE. Pp. 178. Geb. RM. 9.50. 1933. (Walter de Gruyter, Berlin)

This is an attractive little book which I much enjoyed reading. It should prove to be extremely useful to the Applied Mathematician who needs a collection of the most important properties of the functions which occur in the solutions of the partial differential equations of mathematical physics in a compact form, and who does not want to go to the trouble of digging them out of the larger and more abstruse treatises devoted to particular functions. The scope of the book may be gathered from the following list of chapter headings: I, Orthogonal functions (11 pages); II, Bessel functions (53 pages); III, Spherical harmonics (38 pages); IV, Lamé's functions (31 pages); V, Asymptotic series (18 pages); and VI, The Gamma function (14 pages). The first chapter contains a useful epitome of the properties of the functions of Laguerre and Hermite.

The only adverse criticism which I have to make is that the title of the book may prove a little misleading; when I first saw an announcement of it, I anticipated a treatise on series of Legendre functions, series of Bessel functions, and the like, representing either arbitrary analytic functions or arbitrary functions of a real variable; but my disappointment at finding that I was mistaken soon evaporated in the pleasure which I got out of finding how admirably the author had developed his subjects and how far he was able to take the reader without apparent effort.

Special mention must be made of the section on conformal representation of Bessel functions, a topic which I think that Dr. Lense was the first to study, and of which he has given an interesting account.

I suppose that it is to be expected that a modern German work would use the symbol I to denote a Bessel function, but I could have wished that Dr. Lense had followed the example of Prof. Emde who, in the latest edition of *Funktionentafeln mit Formeln und Kurven* (E. Jahnke u. F. Emde), has retained the more usual J.

There are only two errors that I have noticed, the initials attributed to William Thomson (p. 77) and the date of the memoir by Rodrigues (p. 88).

Even at the present rate of exchange, the book is well worth its price.

G. N. W.

Continuous Groups of Transformations. By L. P. EISENHART. Pp. ix, 301. 19s. 6d. 1933. (Princeton University Press; Humphrey Milford)

Besides being, in itself, one of the most attractive branches of mathematics, the theory of continuous groups is demanding more and more attention from specialists in other topics. Compared with other subjects of similar importance there is a remarkable shortage of expository writing on continuous groups, and there was a pressing need for just such a book as the one under review. But apart from its present scarcity value, this book is likely to remain one of the standard works on the subject for many years. In its relation to the subject as a whole, and as an example of craftsmanship, it is similar to Eisenhart's well-known book on Riemannian geometry.

After three preliminary sections on existence theorems, the first chapter introduces the idea of a group as a set of transformations satisfying the usual axioms, and proceeds to the discussion of a group depending on a finite number of parameters. This discussion is not quite accurate, as many sets of transformations which are called groups in this and in other accounts of the Lie Theory do not satisfy the axioms referred to. For an analytic transformation in the space of n real or complex variables does not, in general, transform the whole space into itself but some region X into a region X'. If another transformation carries Y into Y', the resultant is not defined unless

$Y = X'$. Without leading to any further trouble this inaccuracy runs through the whole book.

All functions are assumed to have as many derivatives as are needed, and the term 'continuous group' is used throughout in the sense in which it was used by Lie. This being so, any continuous group determines a set of differential equations, and the main body of the chapter is devoted to certain fundamental theorems about these differential equations, in particular to the so-called 'fundamental theorems' of Lie. The presentation of the third fundamental theorem is open to criticism. After a precise statement of the theorem the chapter concludes by remarking that it has only been proved in a modified form. The latter is amply sufficient for the purposes of the book and the proof given is extremely elegant. But there does exist a proof of the complete theorem, due to E. Cartan, which might have been mentioned explicitly, though a detailed account of it would have been beyond the scope of the book. Incidentally, Cartan's *Mémorial* on continuous groups, which is the only reference given, contains no more than a sketch of the proof without any reference to the two notes (Cartan, *Comptes Rendus*, Vol. 190 (1930), pp. 914 and 1005) where the details are to be found.

The second chapter contains an account of the main theorems about transformation groups operating in a given space. Such theorems are primarily concerned with loci and sets of loci which are transformed into themselves. The chapter gives conditions for their existence and describes the various classifications (transitivity, imprimitivity, etc.) based on such loci. There are sections on the equivalence of two groups, on extended groups and on differential invariants.

The third and fourth chapters are about properties which are common to all groups simply isomorphic with a given one. In the Lie theory one is generally interested not so much in a group as in an infinitesimal group, and the abstract properties of an infinitesimal group depend only on the constants of structure. So the third and fourth chapters are mainly concerned with the algebra of the constants of structure. In Chapter III there is an account of invariant sub-groups, factor-groups, series of composition and integrable groups, and Chapter IV is about the characteristic equation of the adjoint group. The first three chapters, and part of Chapter IV, are confined to material which, except for a few details, is to be found in the writings of Lie. In Chapter IV there appear theorems due to Killing and Cartan, and the last half of the chapter describes the normalization of the constants of structure used in the classification of semi-simple groups. The actual classification is not carried through in detail, but in the last two sections, on semi-simple groups, the normalization is completed and the method of classification is indicated, with references to recent work by H. Weyl, J. A. Schouten (unpublished) and T. S. Graham (unpublished), in which the original methods of Killing and Cartan have been considerably simplified. (Since the appearance of the book B. L. van der Waerden has published a paper—*Math. Zeit.*, Vol. 37 (1933), p. 446—on the determination of the root-figures.)

Chapter V is about the part of the theory which overlaps with differential geometry. After a preliminary section on Riemannian geometry there is an account of the three linear connections, two asymmetric and one symmetric, determined by a simply transitive group, and of the differential geometry of the group manifold. The chapter concludes with an account of motions in Riemannian spaces and in linearly connected spaces.

The final chapter deals with contact transformations and their applications to dynamics and with the closely related 'function-groups' introduced by Lie. Three kinds of contact transformation are discussed in some detail, those described respectively as homogeneous, non-homogeneous and restricted homogeneous contact transformations. Those defined by the condition

$$\bar{p}_i d\bar{x}^i = p_i dx^i + dW(x, p)$$

(the homogeneous kind are defined by this condition with $dW=0$) are, to many people, at least as interesting as any, and their applications to invariant integrals and to dynamics are as beautiful as anything in this kind of mathematics. Here their importance is, I think, underemphasized. They are only introduced implicitly as the transformations induced on the variables $x^2, \ldots, x^n, p_2, \ldots, p_n$ by restricted homogeneous contact transformations, and the fundamental property,

$$d\bar{p}_i\delta\bar{x}^i - \delta\bar{p}_i d\bar{x}^i = dp_i\delta x^i - \delta p_i dx^i,$$

is relegated to an example (p. 281, Ex. 10).

In the final sections on function groups it might have been well to have written a paragraph on the term " function group ". Though any two functions in a function group can be combined by means of Poisson brackets to give a function in the group, the associative law is not satisfied. Therefore a function group does not satisfy the conditions given in the first chapter. Otherwise these sections are admirable.

The opinions stated in the first paragraph of this review can best be supported by dividing the book into three sections. Take first the first three and a half chapters and the last chapter, which are mainly taken up with the part of the theory due originally to Lie himself. The principal theorems are proved clearly and concisely and the arrangement of the whole is coherent. Naturally one cannot say that no one need read anything else about the Lie Theory. There are interesting points of view, geometrical interpretations of formulae and so on, which are not mentioned, and anyone who wants to specialize in the subject must at least glance through many other passages in the literature, forming his own tastes and preferences. But whatever his tastes are, and they may well be completely catered for in this book, he will surely find these chapters invaluable. Having mastered these two hundred pages or so he will feel quite at home among the voluminous writings of Lie.

Similar remarks apply to Chapter V, except that this contains an exposition of more recent work. The material contained in the sections on simply transitive groups and on the group space have not appeared in any previous book.

Many people will value the last half of Chapter IV at least as much as anything else in the book. The analysis of the constants of structure is less formal and more difficult to see whole than anything in the other chapters. The only other published accounts of it are in Cartan's thesis (1894) and in a paper by Weyl (1926). Here the rather complicated algebra involved is presented neatly yet in explicit notation which is easy to follow.

There are numerous examples which are most useful in supplementing the text.

J. H. C. W.

Inversive Geometry. By Frank Morley and F. V. Morley. Pp. xii, 273. 16s. 1933. (Bell)

Readers of the *Gazette* are familiar with Morley's theorem on the trisectors of the angles of a triangle, and those interested in geometry will recall the Morley-Petersen configuration of ten lines and Morley's investigation of Clifford's chain of theorems. All these and much more they will find treated here.

The algebra and analysis of the complex variable is used in the service of geometry ; as the algebraic counterparts of certain planar transformations, translations, rotations, reflections, similitudes and inversions, are very simple in the complex domain, the method has often been applied to particular geometric problems, but nowhere with such generality and completeness as in this book. The reader, turning the pages at random, will strike sections on the theory of three similar figures, on the minimum property of the pedal triangle, on Brocard points, on constructing regular polygons by knotting

bands, he will be surprised by the proof of Feuerbach's theorem, and an extension to space of Hart's theorem, and will probably encounter a number of theorems he has not seen before.

The following has an interest of its own:

Take a set of five circles, with their centres on a given circle, each intersecting the next on the given circle. Join the five other cuts of adjacent circles in order, to obtain a pentagram. Then the five outer vertices of the pentagram lie respectively on the five circles.

Where so much is given, it is ungrateful to ask for more, but I wish a reference had been made to Loud's chain of theorems on incentres. Coolidge's *Circle and Sphere* does not mention them, and there may be a risk of their being overlooked.

Although we have only mentioned details so far, the book is certainly anything but a collection of odds and ends. The general theory is fully dealt with and the particular investigations referred to are given as necessary illustrations. Nor is it merely concerned with "elementary" theorems. Much will be found on the cardioid and deltoid, on lines of flow, on the differential geometry of the complex plane, and on three-bar motion and related topics.

It would be a mistake to suppose that in dealing with geometry by means of the complex variable we are compelled to stay in the plane. Many years ago Klein and H. Wiener pointed out the intimate relations which exist between projective and inversive geometries. Projective geometry of the complex line can be translated into inversive geometry on the plane or sphere; if the sphere be regarded as immersed in three-dimensional space, then from theorems on projective geometry of the line or inversive geometry of the plane we obtain theorems on transformations of three-dimensional space which leave the sphere invariant; then regarding this sphere as the absolute, we reach theorems on metric non-euclidean geometry, and then, by a limit process or otherwise, those in euclidean geometry. For example, the following two theorems correspond:

(i) given two distinct *general* pairs of points on a line, there is just one pair which separates both harmonically;

(ii) given two skew lines in euclidean space, there is just one line cutting both at right angles.

The inner reason why these projective, inversive, and metric geometries correspond so closely, is that their fundamental groups are made up of operations each of which can be expressed, in many ways, as the product of two transformations, whose squares are identity. (Cf. Thomsen's paper in the October number of the *Gazette*, 1933.)

Instead, however, of simply transforming theorems from one geometry to another, we could proceed algebraically and interpret the same algebraic work in different ways corresponding to the different geometries. A point of view, not unlike this, is implied in this book. The roots of a quadratic equation can represent a point-pair on the Argand diagram, or a point-pair on the sphere, and if the sphere is immersed in three-dimensional space, this pair of points can be joined by a line. Then, for instance, the Morley-Petersen configuration connects up with the existence of an orthocentre of a triangle, and with Hesse's theorem on triangles polar with respect to a conic.

The logical purist could pick holes in many statements and definitions, but this would be an easy and useless triumph. "You can't play electromagnetic golf by the rules of centrifugal bumblepuppy", and in the game played here, intuition is freely used. And we must pay a tribute to the technical skill shown in the proofs, to their charm, originality, and uncommon suggestiveness, which invites the reader to do his part.

It would be superfluous to commend this book to geometers, for they will have ordered it by the first post, but all mathematicians will find here a great

deal of amusing and exciting reading—where else, for instance, will the algebraist see the geometric meaning of the Lagrange resolvent? School libraries should buy it, for it would be a real revelation to a gifted sixth-former, and to many a mathematical master. The present reviewer owes already several hours of pure enjoyment to the authors and publishers.

H. G. F.

Adventures of Ideas. By A. N. WHITEHEAD. Pp. xii, 392. 12s. 6d. 1933. (Cambridge)

Professor Whitehead seems to have definitely crossed the line between critical and speculative philosophy. After the long years he spent in the analysis of the fundamental concepts and methods of science, he has some right to embark on an adventure in the realm of speculation. His endeavours to indicate a way of understanding the nature of things and to point out how that way of understanding is illustrated by a survey of the mutations of human experience, resulted in his three now famous books, *Science and the Modern World*, *Process and Reality*, and *Adventures of Ideas*. In the last one of these, Professor Whitehead traces the effects of certain ideas in promoting the slow drift of mankind towards civilisation. With seventy-league boots, he leaps through the centuries of culture, now confronting Egyptian and Greek thought with our own, then interweaving history with philosophy and science, always using the magic of his wit and expression to force upon us striking definitions and brilliant generalisations about sociology, cosmology and philosophy. This very magic, however, has its dangers; for one is apt to stress points which help one's argument, though they might be less important or satisfactory than others which are left unnoticed or ignored.

Obviously Professor Whitehead is more at home with Plato than with any other thinker of the past. He follows the play of Platonic principles through Christianity to our own day, and seems very much impressed by what he calls Plato's supreme discovery, that " the divine element in the world is to be conceived as a persuasive agency, and not a coercive agency " (p. 213). The progress of civilisation is simply the emergence and development of persuasion as the guarantee of world order. This persuasive element finds its highest incarnations for us in Truth, Beauty, Adventure and Peace. Their attainment is not only the justification of the civilised life, but also the key to the understanding of the world, and the essential qualification for the good life. The supreme Truth that Peace demands and Adventure can secure, is Beauty, that is, the conformation of Appearance with Reality.

T. GREENWOOD.

Gruppentheorie und Quantenmechanik. By H. WEYL. 2nd Edition, Pp. xii, 366. Geh. RM. 24. Geb. RM. 26. 1931. (Hirzel, Leipzig)

The first edition of this book, written by one of the foremost mathematicians of our time, about the mathematical background of the most recent theoretical developments in physics, appeared in 1928. At that time there were few indeed able to appreciate the beautiful way in which the abstract ideas of group theory could reduce to order the various seemingly unconnected methods that had been applied by theoretical physicists to particular problems. Unfamiliar as were its methods the book was very difficult to read.

This second edition has been rewritten in such a manner as to simplify the necessary task of its reading for the student of theoretical physics. It can now be followed without any previous knowledge of group theory. In the first chapter the author builds up that " unitary geometry " in terms of which the physical universe is to be described; in the second, he develops the quantum theory and sets up its laws; in the third, he expounds that part of group theory which is required. The fourth chapter contains the simpler applications, of the rotation group, the Lorentz group, and the permutation

group, to determine the general properties that various solutions of particular problems in the quantum mechanics must have. The fifth chapter contains the more complicated theory of the permutation group and its applications; the most beautiful of which, perhaps, is to show that the analogy noticed long ago by Sylvester between the descriptive properties of certain groups and of chemical compounds has a very real background; the wave-equation of the set of particles combining to form a molecule in fact admits just such groups.

The present form of quantum theory depends essentially on the linearity of its equations; and thus might be much modified if that had to be given up. That part of the results, however, which depends simply on the equations admitting the Lorentz group and the permutation group would remain. Weyl's work shows how much it is that would remain. L. H. T.

Gruppenbilder. By W. Threlfall. With 47 Diagrams. Pp. 59. M. 3.30. 1932. (Hirzel)

Most of these group-pictures consist of partitions of a multiply-connected surface into polygons. The operations of a group are represented as automorphisms of a suitably chosen surface; each polygon, or pair of polygons (one "white" and one "black"), is a fundamental region. The case considered most thoroughly is when the polygons are triangles, which combine in sets to form the faces of a "regular polyhedron". Given the connectivity of the surface, the possible numbers of vertices, edges and faces are enumerated by means of the generalization of Euler's Theorem. On page 44, Dr. Threlfall gives an interesting table of the fourteen possibilities for an orientable surface of genus 2. However, for any such numerical solution, there may be a great variety of non-isomorphic groups, corresponding to different ways of fitting the given number of faces together. In order to illustrate the difficulty of enumerating these, Dr. Threlfall devotes his last twelve pages to the comparatively simple case when each of the four vertices belongs to five of the ten edges, while the faces consist of four pentagons.

The treatment is clear and systematic. There are abundant references and attractive diagrams. H. S. M. Coxeter.

The Calculus of Finite Differences. By L. M. Milne-Thomson. Pp. xxiii, 558. 30s. 1933. (Macmillan)

A book on this subject is long over-due. Whittaker and Robinson might have written it some years ago, but they diverged from the path which led to a purely Finite Differences book to wander in the fields of least squares and correlation. Milne-Thomson has written a comprehensive account of most topics which are closely allied to the finite difference notion, and has, for the most part, avoided the temptation to wander down paths which would lead too far afield.

The book is the result of study of two main sources of information and inspiration; the first source is, naturally, Boole; the second source, as inevitable as the first, is the modern work of Nörlund. Such a book is bound to be an invaluable work of reference; in the hands of a bright student I think it will prove an excellent book for study, but I am inclined to think that the less bright student will find the book hard to read unless he is carefully guided by an older and more experienced mathematician. It is the old story of trying to put a treatise and a text-book inside one pair of covers. Given that such a dual task was before him, the author has made a very good job of a difficult undertaking; but that does not prevent the carping reviewer from wishing that the author had written two books, the first one small, elementary, and done expressly for the beginner, the second one larger,

taking much for granted, and written expressly for the mathematician of some attainments.

Let me quote one instance where, in my opinion, the book is admirable from the point of view of the mathematician who knows something of these things before he opens the book, but a trial of unnecessary difficulty to a beginner. The topic of Chapter VI is " The Polynomials of Bernoulli and Euler ". The chapter begins with a very general class of polynomials called " The ϕ Polynomials ". These are then specialised to the β polynomials, and so we pass to the polynomials of Bernoulli and their properties. On the eleventh page of the chapter we meet the " Generating functions of Bernoulli's numbers ", but these are not the familiar B_n ; the familiar B_n appear on the fourteenth page of the chapter under the heading " Bernoulli's numbers of the first order ". There, at last, we find the results which every mathematician learns sooner or later in his career.

There can be no question of the fact that the author has, in Chapter VI, given an admirable account of a field of study which has fascinated him and other mathematicians in recent years. My grouse is that an intelligent reader ought to be given the first things first and their generalisations, however elegant and fascinating, second. Personally, though not an expert in this particular field, I enjoyed Chapter VI ; but I had been familiar with friend Bernoulli for many years and had at least glanced at some of Nörlund's work. To take a parallel case ; the Legendre polynomials are but particular cases of Jacobi's polynomials, but I hope that no teacher will insist that the budding physicist (or even the budding mathematician) be introduced to Legendre *via* a lengthy interview with Jacobi. But Milne-Thomson holds firmly to the opposite view and, whenever possible, his book starts a topic with the most general form it can find.

One consequence of this is that the book makes considerable demands on the skill of compositors and proof-readers ; for example, all the usual apparatus of E and Δ is set out with the minor complications consequent upon working with an interval ω instead of the unit interval. But the printing is excellently done, and both author and publisher are to be congratulated on a piece of difficult printing that has been excellently set, clearly produced, and carefully corrected. W. L. F.

Lezioni di Analisi. I. By F. Severi. Pp. viii, 434. L. 75. 1933. (Zanichelli, Bologna)

Professor Severi is one of the masters of the brilliant Italian school of mathematics, and his present volume, a masterly introduction into the fundamental notions of algebra and analysis, is the result of nearly thirty years of teaching experience. The main feature of this book is rigour combined with simplicity. In the main text we get a coherent and lucid presentation of all the elementary facts of algebra and analysis completed in the Complements and Exercises by modern relevant results.

Thus Chapter 1 on combinations and powers of binomials is complemented by groups of substitutions, probability and finite differences. Chapter 2 together with the Complements is a pretty exhaustive and excellent survey of the theory of determinants and simultaneous linear equations, with matrices, linear substitutions and secular equation thrown in.

Chapters 3 and 4 give the Dedekind cut theory of real numbers in a simplified form and an introduction to complex numbers respectively, completed in the exercises by the elements of sets of points. The reviewer believes that the theory of real numbers given in the first chapter of his book *The Taylor Series* is still simpler than any presentation along the Dedekind cut line can be.

Chapter 5 gives an elegant exposition of limit and allied notions together with a short but fairly complete account of the theory of sets, including

uniform and semi-continuity, and the elements of topology like connection of domains and the invariance of the dimension number. We cannot praise this chapter highly enough.

Chapter 6 on differentiation, including Taylor's theorem, indeterminate forms, maxima and minima, curvature, etc., contains some excellent exercises on the contact of curves.

In the complement to the beautiful Chapter 7 on infinite series all the intricacies of double series have been reduced by the author's new way of approach to some lucid and simple statements. Every teacher of this delicate subject will be grateful to Professor Severi for his splendid help in this part of analysis bristling with subtleties.

The next chapter, on integration, is a meagre one. Obviously we shall get the complete integral calculus in the next volume.

The last chapter, of 123 pages, is an introduction to the algebraic theories fundamental for the understanding of the methods of the Italian school of algebraic geometry. It contains divisibility of polynomials of one and several variables, algebraic functions, fundamental theorem of algebra, symmetric functions, resultant, Bezout's theorem, cubic and quartic equations, calculation of roots, complemented in the small print by Tschirnhausen's transformations, interpolation, prime polynomials in several variables, modern extensions of Bezout's theorem in the language of algebraic geometry in any number of dimensions, together with some recent important results of the author's.

The length and weight of the last chapter makes this beautiful book rather topheavy. Also the splitting of the text into two parts of different character does not make for elegance. An index would have considerably enhanced the value of this work in practical use. We repeat, however, that this volume as it is ranks foremost among the best works on Analysis. P. Dienes.

Integralrechnung. By M. Lindow. 4. Aufl. Pp. 102. RM. 2.40. **Gewöhnliche Differentialgleichungen**. By M. Lindow. Pp. vi, 121. RM. 3. 1933. Mathematisch-Physikalische Bibliothek, Reihe II ; 3, 4. (Teubner)

These books are Nos. 3 and 4 of Teubner's Mathematisch-Physikalische Bibliothek, Reihe II, No. 2 of which is *Differentialrechnung* by the same author. These deal with the elementary parts of integral calculus and differential equations. Each of them contains a large number of interesting illustrative worked examples. These are, in general, taken from many branches of applied mathematics (dynamics, elasticity, magnetism, astronomy, etc.). There is also a good selection of examples for the reader. Both books contain chapters on numerical methods of approximation. The range covered by the second of these books is : elementary properties and methods of integration of equations of the first order, and simple types of equations of the second order. The numerical methods are given for all the types considered.
R. C.

Kotierte Projektionen. By K. Bartel. Translated into German from the second Polish edition (1931 ; first edition 1914) by W. Haack. Pp. vi, 80. Geb. RM. 4.60. 1933. (Teubner)

Here is an excellent introduction to a most attractive branch of descriptive geometry. Except in its applications to civil and mining engineering it has been so neglected in this country that little of its terminology has been standardised. The Figured Plan and Topographical Projection are our nearest equivalents to the title.

Because of this lack of training in the principles and methods of topographical projection, practical men find it necessary, in dealing with problems properly belonging to this field, to convert them into problems in biorthogonal

projection, and thus to surrender them to the other main branch of descriptive geometry. There is no advantage in doing this; it may introduce unnecessary complications, and certainly means the loss of much of the beauty and simplicity which are peculiar to the topographical method. In fact, at least one great continental school places the topographical method, both in time and in importance, ahead of the method of Monge.

This text of Bartel's is admirably straightforward in its methods. The ground covered in its four chapters is: I. Point, Straight Line and Plane; II. Curves in Space, Ruled Surfaces; III. Applications and Examples; IV. Topographical Surfaces. The examples in Chap. III are particularly good; they are definitely practical (*e.g.* passage of a track of uniform gradient from cutting to embankment, lay-out of a level platform and ramp on sloping ground) without being robbed of illustrative value by too great detail. Chapter IV is also good as far as it goes, but should have been made a little more exhaustive.

There are 71 figures, many of them practically full-page. They are splendidly drawn, if, perhaps, occasionally a little over-elaborated. The book is well printed on art paper, which displays the diagrams to the best advantage. There are no examples given to the reader to solve. The concluding bibliography contains 17 items in French, 10 in German, one in Italian and one in Polish.

E. L. I.

Differential Equations for Electrical Engineers. By P. Franklin. Pp. viii, 299. 16s. 6d. 1933. (John Wiley, New York: Chapman & Hall)

"This text is the outgrowth of a course given for over ten years to electrical engineering students in their junior year at the Massachusetts Institute of Technology" (Preface). These students, it would appear, have passed through a first course in the calculus. So it is one more case of the old problem of grafting a few twigs of specialised knowledge on to an immature mathematical stock, and hoping for the best results. It is a problem which will remain unsolved until a very much closer contact is established between pure mathematics and the practical sciences. To-day the branches of science and technology are rather like a loosely-coordinated League of Nations with mathematics standing aloof in splendid isolation. On the other hand, we who claim to be mathematicians, in however humble a capacity, do always find it hard to persuade our practical colleagues to state their difficulties in a form stripped of irrelevant technicalities.

Credit ought therefore to be given, but seldom is, to those who have devoted some of their leisure time to an attempt to harmonise the points of view of the theoretical mathematician and the technical expert. Unfortunately, an attempt to record personal experience in permanent form too often results in a *pot pourri* of information whose title disarms criticism by the qualification "for engineers" or the cruder word "practical".

The tools which the pure mathematician has forged are not necessarily all too delicate to leave his super-sensitive hands. With due admonition as to their proper use, some of them may with advantage be lent to the engineer. And he, surely, is not so coarse a brute that he cannot appreciate the proper use of those instruments of symbolism which, by briefly epitomising his problems, lead to their solution.

Though it is not without blemish, the work now reviewed is an honest attempt to make one branch of pure mathematics, and that the most useful, more readily accessible to practical men. The pity is that such an attempt should be necessary. A logical course in pure mathematics for engineers would lead up naturally and quickly to linear differential equations, ordinary and partial; in actual curricula these are tacked on clumsily at far too late a stage. But, as things are, one must accept such makeshift devices and judge them on the basis of utility, not of idealism.

The title does not truthfully, all-the-truthfully, and nothing-but-the-truthfully, declare the contents of the book. Differential equations occupy little more than one-half of its extent and, with a few exceptions, those considered are linear with constant coefficients. Chapter I (pp. 1-41) and Chapter II (pp. 42-79) are respectively headed "Complex Numbers" and "Average Values and Fourier Series". One is a little surprised that students who have passed through a first course should require initiation into the elementary manipulation which fills the greater part of Chapter I, and wonders how far they can follow the author when he suddenly accelerates his pace and acquires an impetus which carries him into conformal mapping and other matters which are not always very relevant.

The manipulation in these chapters is painfully laboured; for instance, two pages are spent in hammering out the Fourier expansion of $\cos^3 x$. But if some passages in the text soothe the reader by their slow motion, footnotes here and there explode like bombshells. For example, on p. 8, "In elementary trigonometry, or in computational work, 'sin 1' means $\sin 1^\circ = \cdot 0175$, while in advanced mathematics it means $\sin 57\cdot 3^\circ = \cdot 842$. Thus, it is occasionally convenient to write $\sin(1+2^\circ)$ to mean $\sin 1\cdot 035$ or $\sin 59\cdot 3^\circ$".

Throughout the first 90 pages i is the imaginary unit (ambiguously defined). From thence to p. 210, except for an occasional lapse to its former meaning, it becomes *current intensity*, and then to the end it is again the i we know. Opinions on this, and the corresponding misuse of e, are best conveyed to our friend the electrical engineer by word of mouth. The notations $\underline{|\theta}$ and $\overline{|\theta}$ (read "lead angle θ" and "lag angle θ") for $e^{i\theta}$ and $e^{-i\theta}$ respectively, may be accepted with mild protest, but it is questionable if (p. 18) it is convenient to "use degrees . . . so that we write $r\underline{|A^\circ} = re^{iA^\circ}$, the right member being defined by this relation, or by the convention that A° means the number $\pi A/180$".

Those chapters whose existence justifies the title of the book are good in a commonplace fashion, but they also suffer from long-drawn-out manipulation, and over-elaboration of particular examples. Chapter III (pp. 80-113) deals with "Linear Differential Equations with Constant Coefficients" and systems of such equations, with useful elementary discussions of electrical circuits and networks. Chapter IV (pp. 114-155) on "Partial Derivatives and Partial Differential Equations" contains a satisfactory treatment of linear partial differential equations of the first order, reducible equations of the second order and fundamental solutions of

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 y}{\partial x^2} = 0.$$

Chapter V on "The Physical Meaning of Certain Partial Differential Equations" (pp. 156-181) and Chapter VI on "Solutions of Partial Differential Equations satisfying given Boundary Values" (pp. 182-210) are the best in the book. They form a satisfactory introduction to the technique of applying partial differential equations to specific problems like the flow of heat in a rod, or of electric current in a cable. The examples are instructive.

The author's own *apologia* for Chapters VII, headed "Analytic Functions" (pp. 211-248), and VIII, "Convergence of Fourier Series" (pp. 249-280), reads *literatim*: "The last two chapters are of a more theoretical nature, containing demonstrations which justify the earlier calculations with power series and Fourier series. The majority of students, owing to limitations of time and interest, will be content with the initial statement of the results. For the abler students, these chapters will both satisfy their immediate curiosity and skepticism, and enable them to consult more detailed works on analysis". One may express some "skepticism" as to the degree of success

likely to be attained. The subject-matter (limits, continuity, convergence, uniformity, power series, derivatives, integrals, etc.) is reasonably well expounded. The painful elaboration of the earliest chapters has gone; the discussion of Cauchy's theorem is surely the most laconic on record. But it is doubtful if these chapters serve any useful purpose; those electrical engineers who may wish to probe more deeply into the theoretical basis of these matters would do better to turn at once to the texts quoted in the bibliography.

The book has some good points, but it could undoubtedly be improved by cutting out much of the irrelevant matter. As it is, the examples (which are numerous and useful) contain several matters of interest and importance which have been elbowed out of the text by the superfluities. The typography is not quite of the first class, and a number of annoying little errors have escaped correction. E. L. I.

An Introduction to the Computation of Statistics. By Shepherd Dawson. Pp. 192. 10s. 6d. 1933. (University of London Press)

The purpose of this book is to provide workers in various branches of applied statistics with practical methods for computing statistical coefficients and estimating significance. Mathematical derivations, except where elementary and brief, are omitted, the author holding that "the calculation of statistics is merely a matter of arithmetic, and the understanding of their significance requires little more than common sense". One may concur; but "the little more, and how much it is!".

The earlier half of the book is taken up with the tabulation and graphical representation of frequency distributions and the computation of the simpler statistics (in the sense of "statistical coefficients"), such as the parameters of location and dispersion. The presentation is careful and deliberate; illustrative examples are not stinted, and plenty of exercises of a practical and genuine kind are provided. There is little to criticise here beyond an occasional tendency to loose statement. The later part of the book, dealing with correlation and the more recent methods of estimating significance, is less sound; there are definite errors, for example at the top of p. 139, where the "obviously" reveals a serious fallacy.

There are a few omissions. A list of Pearson's types of skew curves is given on p. 80, but the Poisson distribution of rare events, which is as important as the less common Pearson types, does not appear. It might have been introduced at p. 81, with the remark that not merely the symmetrical binomial, but the unsymmetrical also, tends to the normal type, unless the mean np is small. In the computation of the correlation coefficient other methods, better provided with arithmetical checks than the one given, might have been included. Incidentally the list of Pearson curves on p. 80 has been taken from the 1st edition (1906) of Elderton's text-book, the 2nd edition (1927) and the revised list given there at p. 46 having apparently escaped notice.

It is perhaps unreasonable to ask an author to avoid altogether introducing in the earlier part of the book terms which are not defined until a later part. This usage may pass, provided the necessary reference to the later section is given, and provided it is not done too frequently. The reader of the present book will find himself turning forwards and backwards a good deal among the pages, and it might have helped him if instead of the mere indication "Table IX", "Table XXVII" and so on, page references had been given.

Some statements made are either vague or loose; for example (p. 68), "hence if $\sqrt{\beta_1}$ is less than $\sqrt{(6/n)}$ the curve is symmetrical", or again (p. 77), "if p is the probability of an event, and n is the number of cases in a sample, then the number of times the event happens in the sample is np". There are also a few numerical slips, a sprinkling of typographical errors, and some discrepancies; the table on p. 116 does not tally with its description on p. 115,

and Table XXXI in the example of p. 168 is not obtained from Table XXVII (p. 135) in the way described, though what has happened is clear.

In spite of occasional blemishes of this kind the book has merit and will meet a definite need, though a certain amount of marginal annotation may be required. A. C. A.

Leçons sur les progrès récents de la théorie des séries de Dirichlet. By V. Bernstein. Pp. xv, 320. 60 fr. 1933. Collection Borel. (Gauthier-Villars)

According to the author of this book there are three main branches of the theory of Dirichlet series. The classical theory, inspired by Riemann's famous memoir on prime numbers and the work of Dirichlet on the primes in an arithmetic progression, aims more or less consciously at discovering theorems in the theory of numbers by means of Dirichlet series. The second branch, largely a post-war growth, considers Dirichlet series simply as a problem in the theory of functions of a complex variable, without ulterior motives. The third branch, the most recent of all, is H. Bohr's theory of almost periodic functions and is more closely related to the theory of functions of a real variable. It might also be useful to distinguish a fourth branch, in which we are primarily concerned with problems of the convergence and summability of series; but this subject, familiar to English readers through the Cambridge tract by Hardy and Riesz, has many points of contact with the second branch.

This book deals entirely with the second branch of the theory. Suppose we are given a general Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}.$$

What can we say about the analytic function $f(s)$ which is the sum of the series in its region of convergence? How do the singularities of $f(s)$ depend on the coefficients a_n and the exponents λ_n? Problems of this kind have occupied Cramér, Ostrowski, Pólya, the author and many other writers in recent years. Here a connected account of their researches is given. Perhaps the analysis lacks the sensational element of the classical theory, but it is a very interesting complement to it. The author is to be thanked for placing such a difficult subject before the reader in so lucid a manner.

E. C. Titchmarsh.

Les calculs formels des séries de factorielles. By J. Ser. Pp. vii, 98. 20 fr. 1933. (Gauthier-Villars)

This book sets out to study the operators defined by

$$\Delta f(x) = f(x) - f(x+1),$$

and $$\Omega f(x) = \phi(x), \text{ where } \Delta\phi(x) = f(x),$$

and $\phi(0) = 0$, applied to the function

$$X_n = (-1)^{n-1}\binom{x}{n},$$

and to developments of functions in series of such polynomials. It should be observed that the function $\phi(x)$, even when definable, is not uniquely determined, since any periodic function of period unity, which vanishes when $x=0$, can be added, for example, $\sin 2\pi x$. Such periodic functions are omitted by the author " dans le seul but de simplification ", but it is only a skilful choice of operands which prevents this neglect from leading to direct error. Several varieties of coefficients, which would better be regarded as generalized Ber-

noulli's numbers, are introduced. Questions of convergence are intentionally ignored throughout.

The book gives the impression of a series of ingenious devices which are given special application without being pursued to a conclusion. Misprints, undefined notations, *e.g.* Ω', and the absence of explanations make the author's meaning difficult and indeed often impossible to follow.

L. M. MILNE-THOMSON.

Cours d'Algèbre. I.-IV. By R. ESTÈVE and H. MITAULT. Pp. 520. 55 fr. 1933. (Gauthier-Villars)

It may be of interest to English teachers to see what kind of work is done by French children of ages 15-19 for whom these volumes are written.

Vol. I deals with algebraic numbers and expressions, equations of the first degree in x or in x, y, and problems leading to such equations. The treatment of algebraic numbers is careful and logical without being too abstract; it includes expressions like $\sqrt{(9+4\sqrt{5})}$ as well as $(-\frac{8}{15})-(-\frac{11}{18})$, and would be too concise for a preliminary course given at the age that is usual in England, but it is probably intended to consolidate an earlier course. Equations are also discussed with care, and the reader is not allowed to forget that the converse of a true theorem is often false. This applies also to Volume II, from which the examples

$$x+\frac{1}{x}=\frac{1}{x}, \quad \frac{x+b}{2}=\frac{x+10}{a-1}, \quad (x+2)(3-x)\sqrt{(x+1)}=0$$

illustrate cases in which points of detail are not overlooked. Volume II also contains a good chapter on elementary inequalities, and another on graphical representation of simple functions. The examples:

"Solve the simultaneous inequalities $x-y+3<0$, $-2x+5y+10>0$" and "Discuss the variation and graphical representation of $|x|+|x-5|$" may be quoted.

The first 80 pages of Volume III are concerned with ax^2+bx+c and equations, inequalities, graphs, and problems connected therewith. Another section deals with $(ax+b)/(cx+d)$, and there are chapters on arithmetic and geometric progressions and common logarithms.

Vol. IV (Compléments d'algèbre à l'usage de la classe de philosophie de l'enseignement secondaire) is in effect an introduction to calculus. When we say that it begins with a discussion of lim $(3x+7)$ and lim $\{(3x^2+4x-7)/(x-1)\}$ for $x\to 1$, and that ϵ occurs on the first page, it will be understood that the high standard of the earlier volumes is maintained. Chapter I introduces the derived function and methods of calculating it. Chapter II contains applications of the derived function; these are mostly concerned with the variation of functions like $\frac{1}{4}x^4-2x^2-8$, $x(x+2)/(x^2-1)$, and $\sqrt{(25x^2-x^4)}$, and we miss the "maximum and minimum" problems and kinematical applications common to most English text-books. Chapter III introduces the idea of a primitive function, with applications to simple areas. The volume ends, appropriately enough, with a short discussion on the primitives of $1/x$.

There are exercises at the end of each volume, amounting in all to about 800.

A. R.

Elementary Algebra. II. By A. W. SIDDONS and C. T. DALTRY. Pp. viii, 133-372, xxiii-lxxiii. 3s. 6d. Without Answers, 3s. 1934. (Cambridge)

Part I was reviewed in the *Gazette* for December, 1933 (No. 226). Part II carries on the good work. Together they cover the syllabus of several certificate examinations, but omit the elementary series required in others.

400 pages of text make a very full book, but any excision would sacrifice something worth retaining. The exercises are so numerous and of such variety that a suitable selection can be made for any class and any purpose.

Oral exercises carefully composed for class discussion should not only ensure a correct understanding of new ideas as they are introduced, but also guard against the misinterpretations to which pupils are prone. And the explanatory text should keep the inexperienced teacher on right lines.

A conscious appreciation of algebraical form is fostered and the useful idea of degree receives unusual emphasis. It seems not to be widely realised how readily the pre-certificate candidate grasps the idea and how fruitful it is even at this stage. In particular, its value as a check in mechanical algebra and factors cannot easily be overstated. Although the authors have called little attention to this application, they have prepared the ground for the teacher to do so.

The inclusion of simultaneous equations involving (i) 3 unknowns, (ii) 2nd degree terms in one equation, should distinguish the usefulness of the two methods of solution—elimination and substitution.

One of the greatest difficulties of the young pupil is to translate the words of a problem into the algebraical symbols of an equation. There are exercises here on translation and re-translation that should help him to realise how much of his work is translation. And throughout the book a wide experience in the patient study of the difficulties encountered in the classroom is placed at the service of teacher and pupil.

To sum up, the use of this book should give the pupil not only a thorough working knowledge but also a real insight into the nature of elementary algebraical processes. And if examiners would approach their job from the same viewpoint as the authors of this book, teachers of the subject would have little reason to resent the examinations which darken their horizon.

F. C. B.

Higher Certificate Algebra. By W. J. Walker. Pp. viii, 136, 16. 3s. Supplementary pages only, 2s. 1933. (Mills & Boon)

This is mainly a book of examples on theory of quadratics, indeterminate coefficients, partial fractions, permutations, combinations, the binomial theorem for a positive integral index, and the use of a few of the well-known infinite series. The first 62 pages form Part III of the author's *New School Algebra*, and include occasional explanatory paragraphs. Then follows an appendix consisting of supplementary exercises, including many worked-out examples, mainly on the same topics.

The book will be found useful to pupils preparing for Higher Certificate examinations.

W. J. D.

The Groundwork of Geometry, Plane and Solid. By F. M. Marzials. Pp. xiv, 198. Complete, 3s. 6d.; or in three parts, 1s. 6d. each. 1933. (Nelson)

Those teachers who still think that all geometrical reasoning should be strictly deductive from the very beginning will not like this text-book, which is intended to cover the requirements of a school course from the age of 11+ up to the school-leaving examination, and aims at the development of a beginner's understanding and imagination through commonsense reasoning suited to his age and experience. It is written in three parts, each part consisting of nine chapters, and each concluding with some simple solid geometry. The complete work contains 52 theorems and 28 constructions.

In Part I it is assumed as an experimental fact that lines in one plane which are inclined at equal corresponding angles to a transversal are parallel, and it is suggested by a diagram that the same lines could be drawn by using a different transversal and equal angles of a different magnitude. The author is perhaps wise in withholding at this stage as a professional secret the diffi-

culty of proving that if two lines make equal corresponding angles with one transversal then they must also make equal corresponding angles with any other transversal. It is, doubtless, true that a natural growth does not commence with the roots, but just below the surface, and extends both upwards and downwards. But the roots are necessary, and many teachers will think that they need attention within the range of a school course.

The idea of symmetry about an axis is used to develop the properties of an isosceles triangle by assuming as an experimental fact that when a sheet of paper is folded so that A is brought into coincidence with B, then the crease is the perpendicular bisector of AB. The geometry of two isosceles triangles on the same base is then considered, and this leads naturally to the ordinary simple constructions.

Part I also includes some simple properties of the circle and a preliminary treatment of simple areas.

Part II deals with congruent triangles, rectilinear figures and their areas, the two simple loci, the angle-properties and tangent-properties of the circle, the theorem of Pythagoras.

Part III commences with geometrical illustrations of algebraic identities, and thereafter elementary algebra is freely used in applications of Pythagoras' theorem to the circle and triangle. Then the subject-matter deals with ratio and proportion, similar triangles, similar figures and solids. Similar figures are introduced as copies to scale.

The work as a whole has been well designed, and contains many excellent features; but careful revision of details, before a second edition is called for, would make it more acceptable as a text-book. For instance:

Pp. 91 and 170. In stating that two triangles are congruent or similar, the letters do not appear in corresponding order.

P. 54. "$6(3\times 2)$" is intended to express that 6 is obtained from the product 3×2.

P. 103. The angle APB is said "to subtend AB".

P. 117. "Draw EQ parallel to ON" should read "Draw EQ parallel to NO".

P. 143. "P is outside AB" is intended to express that P is in AB produced.

In some cases the proofs given are incorrect or incomplete:

P. 50. It is suggested as a proof that equal chords of a circle must subtend equal angles at the centre because by rotation round the centre equal chords could be made to coincide.

P. 98. The inclusive and exclusive meaning of the word *locus* is clearly stated, but only one aspect is treated in the proofs.

P. 106. Both the enunciation and the proof are incomplete in "If two, or more, equal angles are subtended by the same straight line they lie on a circle of which the straight line is a chord". And on p. 108 there is confusion between this theorem and its converse.

P. 151. The length of AL is taken as "$b-x$", but it may be $x-b$.

Obsolete or unimportant matters might very well be omitted:

P. 88. The "rhomboid" is defined.

P. 121. The unimportant theorem about the equality of the "complements" in a parallelogram is inserted, while that relating to a triangle and a parallelogram on the same base and between that base and a parallel to it is omitted.

Some of the definitions give too much for the construction of the figure:

P. 54. The *rectangle* is introduced as "a four-sided figure with all its angles right-angles".

P. 57. A *right-angled trapezium* is defined as "a four-sided figure with two of its sides parallel and two of its angles right-angles".

With regard to the solid geometry, the pupil, doubtless, perceives by in-

tuition and experience that through every point there exists a straight line which is normal to any given plane, and that normals to the same plane are parallel lines. It will, perhaps, be generally agreed that such assumptions should, at least in the early stages, be accepted and encouraged, without attempting formal proof. But is it wise, after enunciating, as on p. 129, the proposition "If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, it is a normal to the plane which contains them", to prove merely that if OA is perpendicular both to OB and to OC, then, assuming that there exists through O a plane to which OA is normal, that plane can be no other than the plane BOC ?

Part III concludes with a short, but very useful, chapter on simple plans and elevations, the answers to numerical examples, and an index.

W. J. D.

Spherical Analytic Geometry, being an Appendix to Methods of Modern Navigation. By EDWARD J. WILLIS. Pp. 45. 1933. (William Byrd Press, Richmond, Virginia)

Methods of Modern Navigation was reviewed in the *Gazette* some years ago. In this Appendix the author commences with the Equation to a Great Circle, using as coordinates the Longitude and Latitude of a point on the circle. The two constants in the Equation are the Longitude of the point where the Great Circle cuts the Equator and the complement of the angle which the Great Circle makes with the Equator. Denoting the Longitude and Latitude of a point in the Great Circle by x, y and the constants by α, β the equation to the Great Circle is $\tan y = \cot \beta \sin (x - \alpha)$.

The Great Circle is now represented by a sort of Polar figure. A circle of diameter $\cot \beta$ is drawn and a point in the circumference is taken as the origin. A tangent is drawn to the circle at the origin. The length of the radius vector to any other point in the circumference gives the value of $\tan y$, and the angle between this radius vector and the tangent is $x - \alpha$.

If this circle is taken to represent the half of the Great Circle in the Northern Hemisphere, an equal circle on the other side of the tangent will give points on the other half in the Southern Hemisphere.

A diagram is supplied with a series of concentric circles at intervals of one degree, the radius of the one marked $y°$ being $\tan y$.

Longitudes can be read round the edge of the diagram by producing the radius vector. The diagram does not extend beyond 50°, and its use is therefore confined to Latitudes within 50° of the Equator. The reason of this is obvious.

The chief objection to the diagram is due to the fact that all points on the Equator are situated at the origin of the diagram, and a particular point on the Equator can only be indicated by drawing the tangent to the circle representing one of the Great Circles passing through the point. The direction of this tangent gives the Longitude of the point. It follows that all points near the Equator are crowded round the origin. Suggestions are made by the author for dealing with such points, but a Mercator's chart would probably be more satisfactory.

The author uses the diagram for getting a first approximation to the solution of various problems in Spherical Trigonometry and Navigation. One or two of those on Navigation seem to be rather fanciful. He claims that results can be obtained correct to 15′, and if the Latitudes and Declinations involved are within 45° N. and 45° S. this accuracy may be possible, but beam compasses must be used to draw the arcs of circles which represent Great Circles making angles greater than 70° or 75° with the Equator.

To get more accurate results the readings from the diagram are substituted in the equations, and the difference in the calculated values is got by tables.

Six- or sometimes seven-figure tables are used to get this difference. The difference is then treated as a differential, and a slide-rule is used for the numerical calculation of the adjusting differentials. As a rule this gives a result correct to 1′, but the process may be repeated if greater accuracy is required. The method is novel and interesting, and (within the limits stated) in the hands of a good draughtsman gives accurate results.

There are a few misprints and some confusion in the signs of trigonometrical ratios on page 9, but the final result is correct. The more strictly trained mathematician has a horror of taking $\cos 120° = +0{\cdot}5$. R. M. M.

Statics : a Text-book for the Use of the Higher Divisions in Schools and for First-year Students at the Universities. By A. S. Ramsey. Pp. xi, 296. 10s. 6d. 1934. (Cambridge)

When Mr. Ramsey's *Dynamics* appeared I welcomed it as the book for which all teachers of mathematical specialists had been waiting. If the same thing cannot be said of this his companion volume on statics, it is only because there is not the same absence of serious rivals. It certainly may be said that the book demands to be considered carefully by all teachers of specialists.

The explanations are clear. The worked-out examples are numerous (over 100), well chosen and clearly explained ; and there are about 500 examples for solution, nearly all of scholarship standard.

It will probably be useful to give details. The titles of the chapters and the number of examples which they contain are as follows.

Ch. 1. A short introduction.
Ch. 2. Vectors (10).
Ch. 3. Forces at a point (41).
Ch. 4. Moments, Parallel Forces, Couples (43).
Ch. 5. Coplanar Forces (29).
Ch. 6. The solution of problems (57). (This is likely to prove a particularly useful chapter.)
Ch. 7. Bending Moments (15).
Ch. 8. Graphical Statics (30).
Ch. 9. Friction (49).
Ch. 10. Centres of Gravity (44).
Ch. 11. Work and Energy (54).
Ch. 12. Flexible Chains and Strings (39).
Ch. 13. Elasticity (23).
Ch. 14. Forces in three dimensions (38).

This last chapter is not very elaborate ; it does not mention the cylindroid. Among the examples I have found two or three which are also included in the well-known collection by Messrs. Robson and Trimble, but the number thus duplicated seems so small as not to matter.

There seems to me to be only one serious complaint to be made against this volume, taken with the companion volume on dynamics, and that perhaps is not so much a complaint against the author as against the scholarship course in general.

One may search these books in vain for examples such as those worked out in Mr. R. S. Heath's *Elementary Statics*, which " relate almost entirely to practical machines such as boats, bicycles and steam-engines ". This is not because the theory of the screw with friction is too easy for a serious treatise, or because such a discussion of " the mechanical actions which accompany the self-propulsion of wheeled vehicles " as was given in the M.A. report on mechanics is unworthy of the attention of a scholarship candidate. It is because of close adherence to tradition in the subject-matter, even though the treatment is fresh.

As long ago as 1909, Messrs. C. S. Jackson and W. M. Roberts wrote in their preface to *A First Dynamics*:

"There is still perhaps a danger that English text-books on Dynamics may become sharply divided into two classes to which the dyslogistic epithets of 'academic' and 'practical' have been applied, the one for the public school-boy and the candidate for University scholarships, the other for the technical student and the future engineer. We regret the existence of this distinction and have tried to be neither 'academic' nor 'practical'".

Mr. Ramsey may regret the distinction, but his course is definitely "academic".

In spite of this—or indeed I fear perhaps because of it—his book is without doubt of great value for the mathematical specialist who is a candidate for a University scholarship and also for the first-year University student. The teacher of mathematical specialists cannot afford to neglect this volume. He should procure it for his own use, and when he has done so is exceedingly likely to procure it also for the use of his students. C. O. Tuckey.

A First Course in Mechanics. With Answers. Pp. viii, 1-230, xx. **A Second Course in Mechanics.** With Answers. Pp. viii, 231-402, xiv. By W. G. Borchardt. 3s. each. 1933. (Rivingtons)

We have here another course in Elementary Mechanics called forth by the Association's *Mechanics Report*. In this course Statics and Dynamics are treated alternately, gravitational units are introduced first and absolute units kept until the beginning of the second book, while an elementary chapter on the rotation of rigid bodies is included at the end. It is claimed in the preface that in the later chapters "will be found all that is necessary for the Higher Certificate Papers". Considering the elementary nature of the projectile chapter, that v^2/r is only proved for v constant, while the majority of the examples are numerical, a pupil would need to have covered this course very thoroughly and to have done all the harder problems in order to do well in one of the harder Higher Certificate papers.

In the first course on p. 93 it seems hardly necessary to give the usual constant acceleration formulae (in thick type) and at the side of them a special set (also in thick type) for the case $u = 0$. In the second course, in the chapter on centre of gravity, the unsound "strip" proofs are given for the parallelogram and the triangle. v^2/r is proved by an unusual, though satisfactory, method, and the student is encouraged to think of the "centripetal force" and equate it to mv^2/r. This is followed by a brief chapter on S.H.M. and the simple pendulum, and the concluding chapter on the rotation of rigid bodies.

The need of helping the pupil to do examples is kept well to the fore; the diagrams are numerous and clear and of the kind that the pupil must draw himself, and there is an abundance of questions for solution, many of them easy. The bookwork generally is reduced to a minimum, its application being shown by worked-out examples. The teacher who is used to using the older books for teaching Mechanics, but is looking out for ones that are more up to date but at the same time not too different, should find his need satisfied in these two books. J. W. H.

Mathematical Facts and Formulae. By A. S. Percival. Pp. v, 125. 4s. 6d. 1933. (Blackie)

This book gives in a very compressed form notes on selected parts of Mathematics—the selection having apparently been made with the intention of meeting the needs of a specific class of laboratory student.

The algebra dealt with includes approximations, detached coefficients, indeterminate, cubic, quartic and other equations. The analytical geometry consists of a clear explanation of how to reduce a second degree equation to

standard form. The formulae for circular and hyperbolic functions are given and there is a chapter on finite differences and observational equations. In the calculus the ground is covered up to the finding of volumes of revolution and moments of inertia. Most of the differential equations likely to be met with in everyday work are to be found in the twenty-eight pages devoted to this section, while at the end there are tables of integrals, square roots, etc.

Errors in printing are few and unimportant; for instance, on pp. 31 and 32 $\sin(m+n)$ and $\sin A/2$ are wrongly printed, while in the tables for $\log \Gamma(x)$ the following corrections should be noted:

(1·255)......*for* 6854 *read* 6834,

(1·529)......*for* 8274 *read* 8174.

On the whole the author is accurate and easy to read.

Those who make everyday use of the results set out in this book will find it a very useful book to have.

But why did not the author give the comprehensive interpretation of dy/dx as the "measure of the gradient of a line" rather than the more restricted one of $\tan\psi$? In the laboratory where unequal units and oblique axes have constantly to be used, it seems more natural to use the former than the latter. The latter implies rectangular axes and equal units, the former does not. This bias already occurs in most theoretical books, but one does not expect to find it in those intended for the practical man. We also feel that the practical man prefers the treatment of integration based on Riemann's definition—even though it be a nominal one in terms of area—to the exclusive use of the inverse differentiation idea.

To the practical man gradient and area are very real things. V. N.

Vorlesungen über Variationsrechnung. By Oskar Bolza. Pp. ix, 705, 13. Geb. RM. 20. Reprint of 1909 edition. 1933. (Koehlers Antiquarium, Leipzig)

This volume is a reprint of a book originally published in 1909. Except for a list of errata it is unaltered. It is an amplified version of a series of lectures delivered at the University of Chicago which were published in English under the equivalent title.

In the first nine chapters (pp. 1-456) there is an exhaustive account of extremals in two dimensions. The first three chapters (pp. 1-153) are confined to curves defined by non-parametric equations and their main object is to establish necessary and sufficient conditions for a strong minimum. The fourth is a supplementary chapter on functions of real variables and existence theorems for ordinary differential equations, and Chapters V-IX (pp. 189-456) continue with the discussion of extremals, using the parametric representation. Variable end-points and extremals with corners are considered in Chapter VI and VIII respectively, and the discussion is completed with an account of curves providing an absolute minimum. The latter is based on Hilbert's solution of the Dirichlet problem by means of the diagonal procedure.

Chapters X-XII (pp. 457-651) deal with isoperimetric problems and with various analogous problems, Chapter XI being about Euler-Lagrange multipliers. The final chapter (pp. 652-687) is about the extremal surfaces of double integrals. The partial differential equation analogous to the Jacobi differential equation (non-parametric form) is derived from the second variation, and sufficient conditions for a strong minimum are derived by means of a surface integral analogous to Hilbert's line integral.

The chapters are divided into blocks (I-III, IV-V, VI-IX, X-XIII), and at the end of each block there are examples on the chapters in it.

After the results of the last twenty-five years, these being largely based on the book itself, one can naturally find a certain amount to criticise in this exposition. Enough of the material to make the rest seem obvious could now

be compressed into a far shorter volume, which might just as well be written about extremals in n dimensions. Most people who are learning the subject will probably find the original, and shorter, English version more suitable than this longer one. But the latter is still one of the standard works on the subject, as the demand for a second impression shows, and no one who is writing about these matters can afford to ignore it. It is not likely that they will want to do so.

J. H. C. W.

Mathematische Principien der Naturlehre. By SIR ISAAC NEWTON. Translated by Dr. J. P. WOLFERS. Reprint of 1872 edition. Pp. viii, 666. Geb. RM. 20. 1933. (Koehlers Antiquarium, Leipzig)

With Rouse Ball's *Essay on Newton's Principia* and Zeitlinger's *Newton Bibliography* (in the Association's *Newton Memorial Volume*) readily accessible, there is no need for me to do more than congratulate Koehlers on their enterprise in reprinting the only German translation of the *Principia*. Most of the numerous misprints in the original edition of 1872 have been corrected; the type is small but clear, and the book is neatly bound. In addition to the translation, there are 90 pages of notes.

This reprint suggests two reflections. The fact that a German translation did not appear till 1872 probably indicates the length of time for which Latin remained the *lingua franca* of mathematicians. Secondly, in 1942 we shall be celebrating the tercentenary of Newton's birth; may we hope that by then this country will no longer lie under the reproach which is implicit in the words of Professor Gino Loria in the *Gazette* for January, 1915: "With one accord all mathematicians are calling for a really complete edition of Newton's works".

T. A. A. B.

Vorlesungen über Algebra. By G. BAUER. 5th edition, revised by L. BIEBERBACH. Pp. x, 358. Geb. RM. 14. 1933. (Teubner)

The first edition of these lectures by Gustav Bauer (1820-1906) appeared in 1903, and was prepared for the press by Dr. K. Doehlemann, who also was responsible for the second and third editions. The fourth edition, which was published in 1928, had been partly prepared by Doehlemann before his death in that year, but the main task of bringing the book up to date was undertaken by Professor Bieberbach. The changes which he introduced, as well as the general characteristics of the book as a whole, are surveyed in Professor Turnbull's review of the fourth edition in the *Gazette*, XIV (May, 1929), p. 469. The main problem of the volume is the resolution of algebraic equations, and a more lucid or more stimulating account will not easily be found.

The only important change in this new edition is in the treatment of the Galois theory, where an improvement has been effected by the introduction of a method of exposition due to Perron, whose excellent two-volumed *Algebra* covers much the same ground as the present work.

The general appearance and printing are admirable; but one fault noted by Professor Turnbull is still apparent in a few places—ambiguity in a determinant due to faulty spacing of the elements, for example, at the top of p. 137.

T. A. A. B.

The Hollerith and Powers tabulating machines. By L. J. COMRIE. Pp. 48. 2s. 1933. (London)

In this pamphlet the reader who is prepared to do his share in following a concise exposition will find an admirable account of those modern miracles, machines which sort, print and count, written by one of our foremost experts. There is so little that these machines cannot do that we may yet have to take a lesson from *Erewhon*.

T. A. A. B.

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